ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT•

Journal of Functional Analysis 205 (2003) 414–463

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes

Tomoyuki Shirai[a,*,1] and Yoichiro Takahashi[b,2]

[a] *Department of Computational Science, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa 920-1192, Japan*

[b] *Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan*

Received 2 September 2002; accepted 9 April 2003

Communicated by D. Stroock

Abstract

We introduce certain classes of random point fields, including fermion and boson point processes, which are associated with Fredholm determinants of certain integral operators and study some of their basic properties: limit theorems, correlation functions, Palm measures etc. Also we propose a conjecture on an α-analogue of the determinant and permanent.

Keywords: Random field; Point process; Fredholm determinant; Fermion process; Boson process; α-Determinant; Palm measure

1. Introduction

There are two special classes of random point fields or point processes that are associated with determinants and permanents. They are called fermion (or determinantal) point processes and boson point processes [9,23–25]. In the present

*Corresponding author.

E-mail addresses: shirai@kenroku.kanazawa-u.ac.jp (T. Shirai), takahasi@kurims.kyoto-u.ac.jp (Y. Takahashi).

[1] Partially supported by JSPS under the Grant-in-Aid for Scientific Research No.13740057.

[2] Partially supported by JSPS under the Grant-in-Aid for Scientific Research No.13340030.

doi:10.1016/S0022-1236(03)00171-X

paper we reformulate and extend them in terms of their Laplace transforms and study some of the basic properties.

The fermion process has been studied from several points of view since [24,25]. Spohn [39] discussed the Dyson model whose reversible measure is a fermion random field associated with the sine kernel (cf. Example 1.1). A further study was given by Osada [28]. Our first motivation was to give a general framework to such studies [30]. Soshnikov studied the Gaussian fluctuation for fermion point processes in [35,36,38]. Borodin and Olshanski used the fermion point processes to describe and study characters of the infinite-dimensional unitary group $U(\infty)$ [4,5].

It is the Gaussian unitary ensemble (GUE) in random matrix theory that exhibits the character of fermion processes in a natural manner: their Laplace transforms are determinants as well as their densities and correlation functions are (cf. [27,40]). On the other hand, the densities and correlation functions of boson processes are permanents while the Laplace transforms are also related to determinants but given by their reciprocals.

Thus, we are led to the classes of random point fields whose Laplace transforms are given by the powers or inverse powers of determinants. Let Q be the locally finite configuration space over a Polish space R. Given a real number α and a locally trace class integral operator K on an L^2-space $L^2(R,\lambda)$, we seek for the probability measure $\mu_{\alpha,K}$ on Q such that

$$\int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle \xi,f\rangle) = \mathrm{Det}(I+\alpha K_\varphi)^{-1/\alpha} \tag{1.1}$$

for any nonnegative test function f where $\varphi = 1-e^{-f}$, $K_\varphi = \sqrt{\varphi}K\sqrt{\varphi}$ and $\langle \xi, f\rangle = \sum_i f(x_i)$ if $\xi = \sum_i \delta_{x_i} \in Q$.

If such a measure $\mu_{\alpha,K}$ exists, its densities (precisely, the densities of its restriction to the finite configuration space over compact subsets) and correlation functions turn out to be given by the following analogue of the determinant and permanent for a square matrix $A = (a_{ij})_{i,j=1}^n$:

$$\det_\alpha A = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^n a_{i\sigma(i)}, \tag{1.2}$$

where α is a real number, the summation is taken over the symmetric group $\mathfrak{S}_n$, the set of all permutations of $\{1,2,\dots,n\}$, and $\nu(\sigma)$ stands for the number of cycles in σ. This quantity is called the α-permanent by Vere-Jones [41] but we refer to it as α-determinant in the present paper in order to emphasize on the following relationship with the Fredholm determinant for a trace class integral operator J shown in Section 2:

$$\mathrm{Det}(I-\alpha J)^{-1/\alpha} = \sum_{n=0}^\infty \frac{1}{n!}\int_{R^n} \det_\alpha\,(J(x_i,x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1\cdots dx_n). \tag{1.3}$$

The fermion process corresponds to the case $\alpha = -1$ where $\det_{-1} A$ is the usual determinant $\det A$ and the boson process corresponds to the case $\alpha = 1$ where $\det_1 A$ is the permanent per A. Now it is almost obvious that the Poisson point processes are within our framework with $\alpha = 0$. Indeed, taking the limit as $\alpha \to \pm 0$ in (1.1) and (1.2), one finds that $\det_0 A = \prod_i a_{ii}$ and that

$$\int_Q \mu_{0,K}(d\xi) \exp(-\langle \xi, f \rangle)$$
$$= \exp(-\mathrm{Tr}(K_\varphi)) = \exp\left(- \int_R (1 - e^{-f(x)}) K(x,x) \lambda(dx) \right). \tag{1.4}$$

Hence $\mu_{0,K}$ is the Poisson point process with intensity $K(x,x)\lambda(dx)$.

The existence and uniqueness was already studied for $\alpha = -1$ (cf. [30,37]) and it is known that the operator $K(I-K)^{-1}$ plays an important role. The generalization to locally trace class operators and general α's can be done in two ways from K and from $J_\alpha = K(I + \alpha K)^{-1}$. First we start from the operator K.

From now on, for simplicity, we will assume that the space R is locally compact Hausdorff space with countable basis and λ is a nonnegative Radon measure on R, and take continuous functions or bounded measurable functions with compact support as test functions. The space Q is then the space of nonnegative integer-valued Radon measures on R. In particular, Q is a Polish space since it is a closed subset of the space of Radon measures with vague topology. The space Q and R will be endowed with their topological Borel structure.

In below we assume that the Radon measure λ is nonatomic. But one can also consider the case where λ is atomic and obtain almost the same results except for some properties based on the absence of multiple points, such as (1.10) below and (6.29) in Section 6.

Our standing assumption is as follows:

Condition A. (A1) The operator K is a bounded symmetric integral operator on $L^2(R, \lambda)$. Moreover, it is of locally trace class: the restriction $K_\Lambda = P_\Lambda K P_\Lambda$ of K to each compact subset Λ is of trace class where P_Λ stands for the projection operator from $L^2(R, \lambda)$ to the subspace $L^2(\Lambda, \lambda)$.

(A2) The operator K is nonnegative definite. In particular,

$$\mathrm{Spec}(K) \subset [0, \infty). \tag{1.5}$$

If $\alpha < 0$, the operator $I + \alpha K$ is also nonnegative definite so that

$$\mathrm{Spec}(K) \subset [0, -1/\alpha]. \tag{1.6}$$

For simplicity of description, we will assume K to be real symmetric. Statements are the same for the case where K is hermitian (symmetric operator on a complex L^2-space) except for Section 6.4.

Example 1.1. Let $R = \mathbb{R}^1$. Take an integrable even function $\hat{k}$ with values in $[0, 1]$ and let k be its Fourier transform. Define K as the convolution operator on $L^2(\mathbb{R}^1, dx)$ with convolution kernel k. Then $\mathrm{Spec}(K) \subset [0, 1]$ and K satisfies Condition A with $\alpha = -1$. The most interesting example in this class is the sine kernel, $k(x) = \sin \pi x / \pi x$ (cf. Remark 5.8 and Corollary 5.12).

We obtain the following existence and uniqueness theorem under Condition A.

Theorem 1.2. *Let R be a locally compact Hausdorff space with countable basis, λ be a nonnegative, nonatomic Radon measure on R and K be a bounded symmetric integral operator on $L^2(R, \lambda)$. Assume Condition* A *and let $\alpha \in \{2/m;\ m \in \mathbb{N}\} \cup \{-1/m;\ m \in \mathbb{N}\}$. Then there exists a unique probability Borel measure $\mu_{\alpha,K}$ on the configuration space Q such that*

$$\int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I + \alpha K_\varphi)^{-1/\alpha} \tag{1.7}$$

for each nonnegative measurable function f on R with compact support where K_φ stands for the trace class operator defined as

$$K_\varphi(x, y) = \sqrt{\varphi(x)} K(x, y) \sqrt{\varphi(y)} \tag{1.8}$$

and

$$\varphi(x) = 1 - \exp(-f(x)). \tag{1.9}$$

The probability measure $\mu_{\alpha,K}$ has no multiple points:

$$\mu_{\alpha,K}(\xi(\{a\}) \geqslant 2 \text{ for some } a \in R) = 0. \tag{1.10}$$

Moreover, its correlation functions are given by

$$\rho_{n,\alpha,K}(x_1, x_2, \ldots, x_n) = \det_\alpha (K(x_i, x_j))_{i,j=1}^n. \tag{1.11}$$

Theorem 1.2 is a consequence of Theorems 3.6, 4.1 and 6.13.

The generalized binomial distribution gives a toy model of Theorem 1.2. Let R be a one point space, λ be a unit point mass on R and κ be a positive real number. Then the Fredholm determinant is reduced to a number and if $|z|$ is small enough,

$$(1 + \alpha(1 - z)\kappa)^{-1/\alpha} = (1 + \alpha\kappa)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{c^{(n)}(\alpha)}{n!} j_\alpha^n z^n, \tag{1.12}$$

where $c^{(n)}(\alpha) = \prod_{i=0}^{n-1}(1 + i\alpha)$ and $j_\alpha = \kappa(1 + \alpha\kappa)^{-1}$. This series is a probability generating function in z if and only if $\alpha > 0$ or $\alpha = -1/m$ with $m = 1, 2, \ldots$. The probability thus defined is called a generalized binomial distribution. In particular, it is called a negative binomial distribution if $\alpha > 0$ in our notation.

There is another sufficient condition for the existence and uniqueness:

Condition B. (B1) $\alpha > 0$.

(B2) The operator K is a bounded integral operator on $L^2(R, \lambda)$ and the kernel function of the operator $J_\alpha = K(I + \alpha K)^{-1}$ is nonnegative.

Under Condition (B2) the operator $K = J_\alpha(I - \alpha J_\alpha)^{-1}$ also has nonnegative kernel.

Example 1.3. Consider a Markov process on R and assume that its transition semigroup T_t admits a continuous transition probability density with respect to λ. Let $R_\beta = \int_0^\infty e^{-\beta t} T_t \, dt, \beta > 0$, be its resolvent and set $K = R_\beta$. Then, by the resolvent equation one obtains $J_\alpha = R_{\beta+\alpha}$ so that K satisfies Condition B.

The following is an immediate consequence from the proof of Theorem 1.2.

Theorem 1.4. *Let K be a bounded integral operator on* $L^2(R, \lambda)$. *Assume Condition* B. *Then there exists a unique probability Borel measure* $\mu_{\alpha,K}$ *on Q that satisfies* (1.7). *Moreover*, (1.10) *and* (1.11) *hold and* $\mu_{\alpha,K}$ *is infinitely divisible.*

Once Theorems 1.2 and 1.4 are established, it is immediate to see the following generalization of test functions using the estimates stated in Lemma 4.2 and the relation $\mathrm{Det}(I + \alpha K_\varphi) = \mathrm{Det}(I + \alpha\varphi K)$ for nonsymmetric trace class operators φK. Thus, one can consider the characteristic function or the Fourier transform of $\mu_{\alpha,K}$ to prove the central limit theorem (Proposition 5.7).

Theorem 1.5. *Assume Condition* A *with* $\alpha \in \{-1/m;\ m \in \mathbb{N}\} \cup \{2/m;\ m \in \mathbb{N}\}$ *or Condition* B. *Then we have*

$$\int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I + \alpha\varphi K)^{-1/\alpha} \tag{1.13}$$

for any complex-valued bounded measurable function f with compact support provided that $||f||_\infty$ *is sufficiently small.*

The existence and uniqueness theorem can also be proved by starting the operator $J_\alpha = K(I + \alpha K)^{-1}$ (Theorem 6.17) by applying a convergence theorem in forms (Proposition 3.11). Then the random point fields $\mu_{\alpha,K}$ might be regarded as "Gibbs measures" (or random field realizations of Gibbs states, if any) under "α-statistics" as will be discussed in Section 6.5 (cf. [32]). If $\alpha = -1$, they are the usual Gibbs measures and are discussed in detail in lattice cases in Part II [31]. See also [3,21,22] for lattice cases when $\alpha = -1$. The Glauber dynamics for fermion point fields in lattice case is discussed by Yoo and the first named author in [33].

When $R = \mathbb{R}^d$ and K is translation invariant, the basic limit theorems for $\mu_{\alpha,K}$ can be proved rather easily since $\mu_{\alpha,K}$ admits both of the "moment expansion" (Theorem 4.1) and the "cumulant expansion" (Proposition 3.9). We can show the law of large numbers, the central limit theorem and a large deviation result in the present Part I. For instance, we obtain the following large deviation result:

Proposition 1.6. *Let K be a convolution operator with kernel k on $L^2(\mathbb{R}^d)$. Take a nonnegative measurable function f on $\mathbb{R}^d$ with compact support and set $f_N(\cdot) = f(\cdot/N)$. Suppose, in addition, that $\|\alpha K\| \leqslant 1$ when $\alpha > 0$. Then*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f_N \rangle) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dx \Phi_\alpha(\hat{k}(t), f(x)), \tag{1.14}$$

where $\hat{k}$ is the Fourier transform of the kernel k and

$$\Phi_\alpha(\kappa, u) = -\frac{1}{\alpha} \log(1 + \alpha\kappa(1 - e^{-u})), \quad \kappa \geqslant 0, u \geqslant 0. \tag{1.15}$$

This proposition with $\alpha = -1$ is nothing but $\mathbb{R}^d$-version of Szegö's first theorem for Toeplitz matrices where $R = \mathbb{Z}^1$ (cf. [15]). In Part II we also give the $\mathbb{Z}^d$-version. See [31].

The determinantal structure brings us further properties. It might be remarkable that the class of fermion processes is closed under the operation of taking Palm measures.

Theorem 1.7. *If μ is the fermion process associated with operator K, then for λ-almost every x_0 the Palm measure μ^{x_0} coincides with the fermion process associated with the operator K^{x_0} defined by*

$$K^{x_0}(x, y) = \frac{1}{K(x_0, x_0)} \det \begin{pmatrix} K(x, y) & K(x, x_0) \\ K(x_0, y) & K(x_0, x_0) \end{pmatrix} \tag{1.16}$$

whenever $K(x_0, x_0) > 0$.

The Palm measure is a basic concept in point process theory and describes the spacing distribution and this theorem will be proved as Theorem 6.5 and a little more general result is obtained in Corollary 6.6.

Under Condition B, the Palm measure of a boson or boson-like ($\alpha > 0$) process is given by the convolution of itself and some measure (Theorem 6.12).

The boson and boson-like processes can be constructed as a mixture of Poisson processes (or a Cox process) with random intensity obeying χ^2-distributions [9].

Theorem 1.8. *Assume Condition* A. *Let* $X(x), x\in R$ *be a Gaussian random field with mean* 0 *and covariance* $K(x,y)$ *which is continuous and* Π_{X^2} *be a Poisson random field over* R *with random intensity* $X(x)^2\lambda(dx)$. *Then,*

$$E[\Pi_{X^2}(d\xi)] = \mu_{2,K}(d\xi), \tag{1.17}$$

where E *stands for the expectation with respect to the Gaussian random field* $X(x)$.

As a by-product we can prove the existence of the random point field $\mu_{\alpha,K}$ for $\alpha\in\{2/m;\ m\in\mathbb{N}\}$ (Theorem 6.13). This gives another proof to the positivity of permanents of nonnegative definite matrices.

In the final Section 7 we will propose a conjecture on the nonnegativity of $\det_\alpha A$.

Conjecture 1.9. *Let* $0\leqslant\alpha\leqslant 2$. *Then* $\det_\alpha A$ *is nonnegative whenever* A *is a nonnegative definite square matrix.*

Theorems 1.2 and 6.13 turn out to be an affirmative partial answer to the conjecture proved by probabilistic methods. Conversely, if the conjecture is true for some $\alpha>0$, the random field $\mu_{\alpha,K}$ exists for any nonnegative definite K. It seems that our conjecture is closely related to Lieb's conjecture on permanents [13]. If we restrict ourselves to the case $\alpha\in\{1/m;\ m\in\mathbb{N}\}$, the conjecture can also be proved algebraically by using expansion (7.3) of $\det_\alpha A$ by using the immanants.

Finally, we should notice here that the random point field $\mu_{\alpha,K}$ exists whenever $\det_\alpha (J_\alpha(x_i,x_j))_{i,j=1}^n$ are nonnegative even if K is nonsymmetric [14,29].

2. Preliminary

2.1. Properties of trace class operators

First of all, let us recall some basic facts on the trace class operators and fix the notations. Let H be a complex separable Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$. A compact operator T is said to be a trace class operator (or a nuclear operator) if

$$||T||_1 = \mathrm{Tr}(|T|) < \infty, \tag{2.1}$$

where $|T| = \sqrt{T^*T}$. The totality of the trace class operators will be denoted by $\mathscr{I}_1$ and $||T||_1$ is called the trace norm. The trace of T is given by

$$\mathrm{Tr}(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle, \tag{2.2}$$

where $\{e_n\}$ is a complete orthonormal system in H and $\mathrm{Tr}(T)$ does not depend on the choice of $\{e_n\}$. Let $H^{\otimes n} = H\otimes\cdots\otimes H$ be the n-fold tensor product of H and we define an inner product $\langle , \rangle$ on $H^{\otimes n}$ by extending

$$\langle \varphi_1 \otimes \cdots \otimes \varphi_n, \psi_1 \otimes \cdots \otimes \psi_n \rangle = \prod_{i=1}^{n} \langle \varphi_i, \psi_i \rangle \tag{2.3}$$

for $\varphi_i, \psi_i \in H$ $(1 \leqslant i \leqslant n)$.

Let $\mathscr{A}H^{\otimes n}$ be the anti-symmetric subspace of $H^{\otimes n}$. For an operator T on H, we denote

$$\wedge^n(T) = T \otimes \cdots \otimes T|_{\mathscr{A}H^{\otimes n}}. \tag{2.4}$$

We need the following two lemmas which can be found in, for instance, [11,34].

Lemma 2.1. (i) *Let S be a bounded operator and T a trace class operator. Then*

$$\mathrm{Tr}(TS) = \mathrm{Tr}(ST) \tag{2.5}$$

and

$$\mathrm{Tr}(|ST|) \leqslant ||S|| \mathrm{Tr}(|T|). \tag{2.6}$$

Thus, $\mathscr{I}_1$ forms an ideal in the Banach algebra of bounded operators.

(ii) *Let T be a trace class operator on a Hilbert space. Then for each $n \geqslant 1$, the operator $\wedge^n(T)$ is also of trace class and satisfies the following estimate*:

$$|| \wedge^n(T)||_1 \leqslant \frac{1}{n!} ||T||_1^n. \tag{2.7}$$

The Fredholm determinant of $I + T$ is defined by

$$\mathrm{Det}(I+T) = \sum_{n=0}^{\infty} \mathrm{Tr}(\wedge^n(T)). \tag{2.8}$$

If, in addition, S is a bounded operator, then

$$\mathrm{Det}(I+TS) = \mathrm{Det}(I+ST). \tag{2.9}$$

(iii) *If $||T|| < 1$ and $T \in \mathscr{I}_1$, then*

$$\mathrm{Det}(I+T) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \mathrm{Tr}(T^n) \right). \tag{2.10}$$

(iv) *The Fredholm determinant $\mathrm{Det}(I+T)$, as a functional from $\mathscr{I}_1$ to $\mathbb{C}$, is continuously Fréchet differentiable. If $-1 \notin \mathrm{Spec}(T)$ its logarithmic derivative is given by the formula*

$$\delta[\log \mathrm{Det}(I+T)] = \mathrm{Tr}((I+T)^{-1} \delta T). \tag{2.11}$$

Remark 2.2. Let T be a trace class integral operator on $L^2(R, \lambda)$ with continuous kernel $T(x, y)$. If we identify a bounded measurable function θ with the multiplication operator by θ and if we denote the eigenvalues of T by $\{\kappa_i\}_{i\geqslant 1}$ and the corresponding normalized eigenfunctions by $\{\psi_i\}_{i\geqslant 1}$, then the Fredholm determinant can be expressed as

$$\mathrm{Det}(I+\theta T) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{R^n} \prod_{i=1}^{n} \theta(x_i) \det(T(x_i, x_j))_{i,j=1}^{n} \lambda^{\otimes n}(dx_1 \cdots dx_n) \tag{2.12}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1\leqslant i_1 < \cdots < i_n} \prod_{j=1}^{n} \kappa_{i_j} \times \int_{R^n} \prod_{i=1}^{n} \theta(x_i) |\det(\psi_{i_j}(x_k))_{j,k=1}^{n}|^2 \lambda^{\otimes n}(dx_1 \cdots dx_n). \tag{2.13}$$

Note that $\prod_{j=1}^{n} \kappa_{i_j}$ are eigenvalues and

$$(1/n!)^{1/2} \det(\psi_{i_j}(x_k))_{j,k=1}^{n} \tag{2.14}$$

are the normalized eigenfunctions of the trace class operator $\wedge^n(T)$ considered as an integral operator on $L^2(R^n, \lambda^{\otimes n})$. These functions (possibly, without the normalizing constant $1/(n!)^{1/2}$) are called Slater determinants in physical literature.

Remark 2.3. The well definedness of the Fredholm determinant appeared in (1.7) is guaranteed by the minimax principle. Indeed, if T is a trace class symmetric operator on the space $L^2(R, \lambda)$ with $\mathrm{Spec}(T) \subset [0, \infty)$ and ψ is a measurable function on R with values in $[0, 1]$. Then the operator

$$T_\psi = \sqrt{\psi} T \sqrt{\psi} \tag{2.15}$$

is also a trace class operator and, for each $k \geqslant 1$, the kth eigenvalue of T_ψ is dominated by the kth eigenvalue of T.

2.2. *Expansion of* $\mathrm{Det}(I-\alpha J)^{-1/\alpha}$

The next theorem is a generalization of (2.12) in Remark 2.2, which is obtained in [41] for finite matrices.

Theorem 2.4. *Let J be a trace class integral operator. If $\|\alpha J\| < 1$, we have*

$$\mathrm{Det}(I-\alpha J)^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{R^n} \det_\alpha (J(x_i, x_j))_{i,j=1}^{n} \lambda^{\otimes n}(dx_1 \cdots dx_n), \tag{2.16}$$

where $\det_\alpha$ *is defined by* (1.2). *If* $\alpha \in \{-1/m;\ m \in \mathbb{N}\}$, (2.16) *holds without condition* $||\alpha J|| < 1$.

Proof. Let $T = \alpha J$. If $||T|| < 1$ we know (2.10) holds. Expanding the exponential in (2.10) of Lemma 2.1(iii), we obtain for any $\beta \in \mathbb{R}$

$$\begin{aligned}\operatorname{Det}(I-T)^{-\beta} &= 1 + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} \sum_{n_1,\dots,n_k \geqslant 1} \frac{\operatorname{Tr}(T^{n_1})\cdots\operatorname{Tr}(T^{n_k})}{n_1\cdots n_k} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\beta^k}{k!} \sum_{\substack{n_1,\dots,n_k \geqslant 1,\\ n_1+\cdots+n_k=n}} \frac{\operatorname{Tr}(T^{n_1})\cdots\operatorname{Tr}(T^{n_k})}{n_1\cdots n_k}. \end{aligned} \tag{2.17}$$

It is well known that there is one to one correspondence between conjugacy classes of the symmetric group $\mathfrak{S}_n$ and partitions of n, that is, $(j_1, \dots, j_k)$ with $\sum_{i=1}^{k} j_i = n$ and $j_1 \geqslant \cdots \geqslant j_k \geqslant 1$. Indeed, the conjugacy class $[\sigma]$ of a permutation is determined by the length j_i $(1 \leqslant i \leqslant \nu(\sigma))$ of cycles in σ. It is easy to see that

$$\frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geqslant 1,\\ (n_1^*,\dots,n_k^*)=(j_1,\dots,j_k)}} \frac{n!}{n_1\cdots n_k} = \sum_{\substack{\sigma \in \mathfrak{S}_n,\\ [\sigma]=(j_1,\dots,j_k)}} 1, \tag{2.18}$$

where $(n_1^*, \dots, n_k^*)$ is the rearrangement of $(n_1, \dots, n_k)$ so that $n_1^* \geqslant \cdots \geqslant n_k^*$. Hence we obtain

$$\begin{aligned}&1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\beta^k}{k!} \sum_{\substack{n_1,\dots,n_k \geqslant 1,\\ n_1+\cdots+n_k=n}} \frac{\operatorname{Tr}(T^{n_1})\cdots\operatorname{Tr}(T^{n_k})}{n_1\cdots n_k} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\beta^k}{n!} \sum_{\substack{j_1 \geqslant \cdots \geqslant j_k \geqslant 1\\ j_1+\cdots+j_k=n}} \sum_{\substack{\sigma \in \mathfrak{S}_n,\\ [\sigma]=(j_1,\dots,j_k)}} \operatorname{Tr}(T^{j_1})\cdots\operatorname{Tr}(T^{j_k}) \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \sum_{\substack{\sigma \in \mathfrak{S}_n\\ \nu(\sigma)=k}} \beta^{\nu(\sigma)} \int_{R^n} \prod_{i=1}^{n} T(x_i, x_{\sigma(i)}) \lambda^{\otimes n}(dx_1\cdots dx_n) \end{aligned} \tag{2.19}$$

and then

$$\begin{aligned}&\operatorname{Det}(I-T)^{-\beta} \\ &\quad = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \beta^{\nu(\sigma)} \int_{R^n} \prod_{i=1}^{n} T(x_i, x_{\sigma(i)}) \lambda^{\otimes n}(dx_1\cdots dx_n). \end{aligned} \tag{2.20}$$

The formal computation as above can be immediately justified if $T \in \mathcal{I}_1$ and $||T|| < 1$. Consequently, we obtain (2.16) by setting $\beta = 1/\alpha$ and $T = \alpha J$. $\square$

If $\alpha = -1$, formula (2.16) is nothing but formula (2.12) if one expresses the traces as the integrals of usual determinants. Thus it is analytic in T and so we can remove the condition $||J|| < 1$ if $\alpha = -1$. More generally, since the left-hand side of (2.16) is the mth power of an analytic function in J, we can remove the condition $||\alpha J|| < 1$ if $\alpha \in \{-1/m;\ m \in \mathbb{N}\}$.

Remark 2.5. Let A be an n by n nonnegative definite matrix. As is well known (cf. [2]), there hold the inequalities

$$\operatorname{per} A \geqslant \prod_{i=1}^{n} a_{ii} \geqslant \det A \geqslant 0. \tag{2.21}$$

In other words,

$$\det{}_1 A \geqslant \det{}_0 A \geqslant \det{}_{-1} A \geqslant 0. \tag{2.22}$$

3. Existence and general property

In this section we will prove Theorem 1.2 for $\alpha \in \{\pm 1/m;\ m \in \mathbb{N}\}$ except for assertion (1.11) on correlation functions which will be proved separately in the next section. The rest cases $\alpha \in \{2/m;\ m \in \mathbb{N}\}$ will be treated in Section 6 by a constructive method.

3.1. Some lemmas

We assume Condition A and, in addition, we assume the following operators are well defined as bounded operators for compact subsets Λ if $\alpha < 0$.

$$J_\alpha[\Lambda] = (I + \alpha K_\Lambda)^{-1} K_\Lambda. \tag{3.1}$$

The operator $J_\alpha[\Lambda]$ is the quasi-inverse of K_Λ in the sense that

$$(I + \alpha K_\Lambda)(I - \alpha J_\alpha[\Lambda]) = I \tag{3.2}$$

(though the terminology is usually used only for $\alpha = 1$). If Λ is compact, the operator $J_\alpha[\Lambda]$ is also a trace class operator with spectrum in $[0, \infty)$. Moreover,

$$\operatorname{Spec}(J_\alpha[\Lambda]) \subset [0, \alpha^{-1}) \quad \text{if } \alpha > 0 \tag{3.3}$$

and

$$\operatorname{Spec}(J_\alpha[\Lambda]) \subset [0, \infty) \qquad \text{if } \alpha < 0. \tag{3.4}$$

Note that $J_\alpha[\Lambda]$ is not a restriction operator while K_Λ is.

Lemma 3.1. *Let Λ be a compact subset of R and $f : R \to [0, \infty)$ be measurable and assume*

$$\operatorname{supp} f \subset \Lambda. \tag{3.5}$$

Then,

$$\operatorname{Det}(I + \alpha K_\varphi)^{-1/\alpha} = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \operatorname{Det}(I - \alpha (J_\alpha[\Lambda])_{e^{-f}})^{-1/\alpha}, \tag{3.6}$$

where $(J_\alpha[\Lambda])_{e^{-f}} = e^{-f/2} J_\alpha[\Lambda] e^{-f/2}$.

Proof. By using (2.9) we can compute the Fredholm determinant as follows:

$$\begin{aligned}
\operatorname{Det}(I + \alpha K_\varphi) &= \operatorname{Det}(I + \alpha K_\Lambda \varphi) = \operatorname{Det}(I + \alpha K_\Lambda - \alpha K_\Lambda e^{-f}) \\
&= \operatorname{Det}(I + \alpha K_\Lambda) \operatorname{Det}(I - (I + \alpha K_\Lambda)^{-1} \alpha K_\Lambda e^{-f}) \\
&= \operatorname{Det}(I + \alpha K_\Lambda) \operatorname{Det}(I - \alpha J_\alpha[\Lambda] e^{-f}) \\
&= \operatorname{Det}(I + \alpha K_\Lambda) \operatorname{Det}(I - \alpha (J_\alpha[\Lambda])_{e^{-f}}).
\end{aligned} \tag{3.7}$$

Hence we obtain the lemma. □

Now let $Q(\Lambda)$ be the configuration space over Λ. If Λ is compact, $Q(\Lambda)$ will be identified with $\bigcup_{n=0}^\infty \Lambda^n / \sim$ where the equivalence relation $\sim$ is defined by permutations of coordinates. Using $\det_\alpha$ we can define a symmetric function $\sigma_{\Lambda,\alpha,K}$ on $\bigcup_{n=0}^\infty \Lambda^n$ as follows: set, for $n \geqslant 1$,

$$\sigma_{\Lambda,\alpha,K}(x_1, \ldots, x_n) = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \det{}_\alpha(J_\alpha[\Lambda](x_i, x_j))_{i,j=1}^n \quad \text{on } \Lambda^n \tag{3.8}$$

and for $n = 0$ if we denote the empty configuration by $\emptyset$,

$$\sigma_{\Lambda,\alpha,K}(\emptyset) = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \quad \text{on } \Lambda^0 = \{\emptyset\}. \tag{3.9}$$

Define a (possibly, signed) measure $\mu_{\Lambda,\alpha,K}$ on $Q(\Lambda)$ by

$$\begin{aligned}
&\int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle) \\
&\quad = \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \sigma_{\Lambda,\alpha,K}(x_1, \ldots, x_n) \exp\left(-\sum_{k=1}^n f(x_k)\right) \lambda^{\otimes n}(dx_1 \cdots dx_n).
\end{aligned} \tag{3.10}$$

The measure $\mu_{\Lambda,\alpha,K}$ will turn out to be a probability measure for $\alpha = \pm 1/m$ in Lemma 3.3 below and for $\alpha = 2/m$ in Section 6. The rest case is to be posed as Conjecture 7.1 in Section 7.

Lemma 3.2. *Let f be a nonnegative measurable function on R. Assume* (3.5) *holds, i.e.,*

$$\operatorname{supp} f \subset \Lambda, \tag{3.11}$$

and set $\varphi = 1 - e^{-f}$. *Then for* $\alpha \in \{-1/m;\ m \in \mathbb{N}\} \cup (0, \infty)$,

$$\int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle) = \operatorname{Det}(I + \alpha K_\varphi)^{-1/\alpha}. \tag{3.12}$$

Proof. Assume $\operatorname{supp} f \subset \Lambda$. If $\alpha > 0$, $||\alpha J_\alpha[\Lambda]|| = ||\alpha K_\Lambda (I + \alpha K_\Lambda)^{-1}|| < 1$ and otherwise we assume $\alpha \in \{-1/m;\ m \in \mathbb{N}\}$. Then we can apply Theorem 2.4 to the right-hand side of (3.12) and we get

$$\begin{aligned}
&\operatorname{Det}(I + \alpha K_\varphi)^{-1/\alpha} \\
&\quad = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \operatorname{Det}(I - \alpha (J_\alpha[\Lambda])_{e^{-f}})^{-1/\alpha} \\
&\quad = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \left\{ 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \operatorname{det}_\alpha((J_\alpha[\Lambda])_{e^{-f}}(x_i, x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1 \cdots dx_n) \right\} \\
&\quad = \operatorname{Det}(I + \alpha K_\Lambda)^{-1/\alpha} \\
&\qquad \times \left\{ 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \operatorname{det}_\alpha(J_\alpha[\Lambda](x_i, x_j))_{i,j=1}^n \exp\left(-\sum_{k=1}^n f(x_k) \right) \lambda^{\otimes n}(dx_1 \cdots dx_n) \right\} \\
&\quad = \sum_{n=0}^\infty \frac{1}{n!} \int_{\Lambda^n} \sigma_{\Lambda,\alpha,K}(x_1, \ldots, x_n) \exp\left(-\sum_{k=1}^n f(x_k) \right) \lambda^{\otimes n}(dx_1 \cdots dx_n) \\
&\quad = \int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle). \qquad \square
\end{aligned}$$

Lemma 3.3. *If* $\alpha \in \{\pm 1/m;\ m \in \mathbb{N}\}$ *the measure* $\mu_{\Lambda,\alpha,K}$ *is a probability measure on* $Q(\Lambda)$ *and* $\sigma_{\Lambda,\alpha,K}$ *is its density with respect to* $\oplus_{n=0}^\infty \lambda^{\otimes n}$.

Proof. If $\alpha = \pm 1$, then it is obvious that the function σ_Λ is nonnegative since $\operatorname{det}_{-1} = \det$ and $\operatorname{det}_1 = \operatorname{per}$ (see Remark 2.5). Hence, $\mu_{\Lambda,\pm 1,K}$ is a probability measure and $\sigma_{\Lambda,\pm 1,K}$ is its density.

By the definition of their Laplace transforms, the measure $\mu_{\Lambda,\alpha/m,K}$ is the m-fold convolution of $\mu_{\Lambda,\alpha,K/m}$:

$$\begin{aligned}
&\int_{Q(\Lambda)} \mu_{\Lambda,\alpha/m,K}(d\xi) e^{-\langle \xi, f \rangle} \\
&\quad = \int_{Q(\Lambda) \times \cdots \times Q(\Lambda)} \mu_{\Lambda,\alpha,K/m}(d\xi_1) \cdots \mu_{\Lambda,\alpha,K/m}(d\xi_m) e^{-\langle \xi_1 + \cdots + \xi_m, f \rangle}.
\end{aligned} \tag{3.13}$$

Hence, $\mu_{\Lambda,\pm 1/m,K}$ is also a probability measure and $\sigma_{\Lambda,\pm 1/m,K}$ is necessarily nonnegative. $\square$

Let $\alpha = -1$ and Λ be a compact subset of R. If the restricted operator K_Λ admits 1 as its eigenvalues, $J_{-1}[\Lambda]$ loses its meaning. So we cannot follow the argument above. But this gap will be compensated for by the next lemma (see also [37]). Thus we may safely abuse notation (3.8) even in the degenerated cases where $\det(I - K_\Lambda) = 0$: the precise definition (3.8) is then given by (3.16) below.

Lemma 3.4. *Let $\alpha = -1$ and Λ be a compact set of R. Let $1 \geqslant \kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant 0$ are the eigenvalues of K_Λ and $\{\psi_i\}_{i\geqslant 1}$ be the corresponding normalized eigenfunctions.*

(i) *Assume that all the eigenvalues of K_Λ are strictly less than 1. Then the density function $\sigma_{\Lambda,-1,K}$ defined in (3.8) can be expressed as*

$$\sigma_{\Lambda,-1,K}(x_1, \dots, x_n) = \sum_{1\leqslant i_1<\cdots<i_n} \left(\prod_{j=1}^{n} \kappa_{i_j} \prod_{k\neq i_1,\dots,i_n} (1-\kappa_k) \right) |\det(\psi_{i_j}(x_k))_{j,k=1}^{n}|^2 \quad \text{on } \Lambda^n. \tag{3.14}$$

(ii) *Assume that 1 is an eigenvalue of K_Λ with multiplicity m. Then there exists a unique probability measure $\mu_{\Lambda,-1,K}$ such that*

$$\int_{Q(\Lambda)} \mu_{\Lambda,-1,K}(d\xi) e^{-\langle \xi, f\rangle} = \mathrm{Det}(I - K_\varphi), \tag{3.15}$$

where $\varphi = 1 - e^{-f}$. Its density function $\sigma_{\Lambda,-1,K}$ is given by

$$\sigma_{\Lambda,-1,K}(x_1, \dots, x_n) = \sum_{\substack{1\leqslant i_1<\cdots<i_n \\ i_1=1,\dots,i_m=m}} \left(\prod_{j=m+1}^{n} \kappa_{i_j} \prod_{k\neq i_1,\dots,i_n} (1-\kappa_k) \right) |\det(\psi_{i_j}(x_k))_{j,k=1}^{n}|^2 \quad \text{on } \Lambda^n \tag{3.16}$$

for $n \geqslant m$ and $\sigma_{\Lambda,-1,K}(x_1, \dots, x_n) = 0$ on Λ^n for $n < m$. In particular,

$$\mu_{\Lambda,-1,K}(\xi(\Lambda) \geqslant m) = 1. \tag{3.17}$$

Similarly, if a positive integer k is an eigenvalue of K_Λ with multiplicity m, then for $\alpha = -1/k$

$$\mu_{\Lambda,-1/k,K}(\xi(\Lambda) \geqslant mk) = 1. \tag{3.18}$$

Proof. (i) Recall that

$$\mathrm{Det}(I - K_\varphi) = \mathrm{Det}(I - K_\Lambda)\, \mathrm{Det}(I + e^{-f} J_{-1}[\Lambda]) \tag{3.19}$$

for any nonnegative measurable function f with $\operatorname{supp} f \subset \Lambda$. Applying (2.13) of Remark 2.2 to (3.19) with $\theta = e^{-f}$ and $T = J_{-1}[\Lambda]$, we obtain

$$\begin{aligned}
&\operatorname{Det}(I - K_\varphi)\\
&= \operatorname{Det}(I - K_\Lambda) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1 < \cdots < i_n} \prod_{j=1}^{n} \kappa_{i_j}(1-\kappa_{i_j})^{-1}\\
&\quad \times \int_{\Lambda^n} e^{-\sum_{i=1}^n f(x_i)} |\det(\psi_{i_j}(x_k))_{j,k=1}^n|^2 \lambda^{\otimes n}(dx_1 \cdots dx_n)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1 < \cdots < i_n} \prod_{j=1}^{n} \kappa_{i_j} \prod_{k \notin i_1, \ldots, i_n} (1-\kappa_k)\\
&\quad \times \int_{\Lambda^n} e^{-\sum_{i=1}^n f(x_i)} |\det(\psi_{i_j}(x_k))_{j,k=1}^n|^2 \lambda^{\otimes n}(dx_1 \cdots dx_n).
\end{aligned} \tag{3.20}$$

(ii) Let $0 < s < 1$ and consider the operator sK. Then one can obtain the probability measures $\mu_{\Lambda,-1,sK}(0 < s < 1)$. On one hand, the Laplace transform $\operatorname{Det}(I - (sK)_\varphi)$ of $\mu_{\Lambda,-1,sK}$ converges to $\operatorname{Det}(I - K_\varphi)$ as $s \to 1$ for any nonnegative measurable function f with $\operatorname{supp} f \subset \Lambda$. Since the Laplace transform determines a probability measure uniquely, we obtain a unique probability measure on $Q(\Lambda)$ associated with K, say $\mu_{\Lambda,-1,K}$.

On the other hand, the probability measure $\mu_{\Lambda,-1,sK}$ has the density function $\sigma_{\Lambda,-1,sK}$ given by (3.14) with $s\kappa_i$ in place of κ_i. Thus taking the limit $s \to 1$, we easily obtain the density function $\sigma_{\Lambda,-1,K}$ of the form (3.16). $\square$

Finally, we note the following fact.

Remark 3.5. Let $\alpha = -1$ and assume Condition A on K. If 1 is an eigenvalue of K_Λ, then 1 is also an eigenvalue of K and any corresponding eigenfunctions are localized on the set Λ. In fact, let $K_\Lambda f_\Lambda = f_\Lambda$ and define $f : R \to \mathbb{C}$ by setting $f = f_\Lambda$ on Λ and $f = 0$ outside Λ. Then

$$\|f\|^2 = \|f_\Lambda\|^2 = \|K_\Lambda f_\Lambda\|^2 \leqslant \|K_\Lambda f_\Lambda\|^2 + \|K_{\Lambda^c\Lambda} f_\Lambda\|^2 = \|Kf\|^2 \leqslant \|f\|^2, \tag{3.21}$$

where $K_{\Lambda^c\Lambda}$ stands for the operator $P_{\Lambda^c} K P_\Lambda$. Hence, $K_{\Lambda^c\Lambda} f_\Lambda = 0$ and $Kf = f$.

3.2. *The existence and uniqueness theorem under Condition* A *for* $\alpha = \pm 1/m$

The existence and uniqueness theorem under Condition B will be treated in Section 6.

Theorem 3.6. *Assume Condition* A *and* $\alpha \in \{\pm 1/m;\ m \in \mathbb{N}\}$.

(i) *The family* $\{\mu_{\Lambda,\alpha,K}; \Lambda \subset R, \text{compact}\}$ *satisfies the Kolmogorov consistency condition and, hence, there exists a unique probability measure* $\mu_{\alpha,K}$ *on the whole* *configuration space* $Q = Q(R)$ *satisfying*

$$\int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) = \mathrm{Det}(I + \alpha K_\varphi)^{-1/\alpha}. \tag{3.22}$$

(ii) *If* $\mathrm{supp} f \subset \Lambda$, *then*

$$\begin{aligned}\int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) &= \int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_n)\exp\left(-\sum_{k=1}^{n} f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n).\end{aligned} \tag{3.23}$$

(iii) *Assume, in addition,* λ *is a nonatomic Radon measure. Then the measure* $\mu_{\alpha,K}$ *has no multiple points*:

$$\mu_{\alpha,K}(\xi \in Q;\ \ \xi(\{a\}) \geqslant 2 \textit{ for some } a \in R) = 0. \tag{3.24}$$

Proof. Let

$$\Lambda_0 = \mathrm{supp} f, \quad \Lambda_0 \cap \Lambda_1 = \emptyset \tag{3.25}$$

and set

$$\Lambda = \Lambda_0 \cup \Lambda_1. \tag{3.26}$$

Then, since $\mathrm{supp} f \subset \Lambda$, we have

$$\begin{aligned}&\mathrm{Det}(I + \alpha K_\varphi)^{-1/\alpha} \\ &\quad = \int_{Q(\Lambda)} \mu_{\Lambda,\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) \\ &\quad = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_n)\exp\left(-\sum_{k=1}^{n} f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n) \\ &\quad = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{1}{m!}\int_{\Lambda_0^m}\int_{\Lambda_1^\ell}\sigma_{\Lambda,\alpha,K}(x_1,\ldots,x_m,y_1,\ldots,y_\ell)\lambda^{\otimes \ell}(dy_1\cdots dy_\ell) \\ &\qquad \times \exp\left(-\sum_{k=1}^{m} f(x_k)\right)\lambda^{\otimes m}(dx_1\cdots dx_m).\end{aligned} \tag{3.27}$$

On the other hand,

$$\begin{aligned}\mathrm{Det}(I+\alpha K_\varphi)^{-1/\alpha} &= \int_{Q(\Lambda_0)} \mu_{\Lambda_0,\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda_0^n}\sigma_{\Lambda_0,\alpha,K}(x_1,\dots,x_n)\\ &\quad\times\exp\left(-\sum_{k=1}^{n} f(x_k)\right)\lambda^{\otimes n}(dx_1\cdots dx_n). \end{aligned}\tag{3.28}$$

Consequently, comparing the above two Eqs. (3.27) and (3.28), one can conclude

$$\sigma_{\Lambda_0,\alpha,K}(x_1,\dots,x_m)=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\int_{\Lambda_1^\ell}\sigma_{\Lambda,\alpha,K}(x_1,\dots,x_m,y_1,\dots,y_\ell)\lambda^{\otimes\ell}(dy_1\cdots dy_\ell),\tag{3.29}$$

which is nothing but the desired consistency condition. Hence by a version of Kolmogorov's extension theorem (e.g. cf. [18]), there exists a unique probability measure $\mu=\mu_{\alpha,K}$ on $Q=Q(R)$ which satisfies

$$\int_Q F(\xi)\mu_{\alpha,K}(d\xi)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda,\alpha,K}(x_1,\dots,x_n)F\left(\sum_{i=1}^{n}\delta_{x_i}\right)\lambda^{\otimes n}(dx_1\cdots dx_n)\tag{3.30}$$

for any bounded measurable function F such that $F(\xi)=F(\xi_\Lambda)$ where ξ_Λ is the restriction of ξ to Λ. Hence we obtain (i). In particular, putting $F(\xi)=\exp(-\langle\xi,f\rangle)$ for f supported by Λ, we obtain (ii).

To prove (iii), it is sufficient to show that for any compact set $A\subset R$

$$\mu_{\alpha,K}(\xi\in Q;\ \xi(\{a\})\geqslant 2 \text{ for some } a\in A)=0\tag{3.31}$$

or, a fortiori, that

$$\mu_{\alpha,K}(\xi\in Q;\ \xi(\Lambda)\geqslant 2)=o(\lambda_1(\Lambda))\tag{3.32}$$

as $\lambda_1(\Lambda)\to 0$ uniformly in $\Lambda\subset A$ where $\lambda_1(\Lambda)=\int_Q\mu_{\alpha,K}(d\xi)\xi(\Lambda)$. However, by using a mean value theorem for the function $g(t)=\mathrm{Det}(I+t\alpha K_\Lambda)^{-1/\alpha}$, we obtain

$$\begin{aligned}\mu_{\alpha,K}(\xi(\Lambda)\geqslant 2) &= 1-\mu_{\alpha,K}(\xi(\Lambda)=0)-\mu_{\alpha,K}(\xi(\Lambda)=1)\\ &= 1-\mathrm{Det}(I+\alpha K_\Lambda)^{-1/\alpha}-\mathrm{Det}(I+\alpha K_\Lambda)^{-1/\alpha}\mathrm{Tr}(J_\alpha[\Lambda])\\ &\leqslant \frac{1}{2}\mathrm{Det}(I+t\alpha K_\Lambda)^{-1/\alpha}|(1+\alpha)(\mathrm{Tr}(J_{t\alpha}[\Lambda]))^2-2\alpha\,\mathrm{Tr}(\wedge^2 J_{t\alpha}[\Lambda])|\\ &\leqslant \frac{1+2|\alpha|}{2}||J_{t\alpha}[\Lambda]||_1^2\end{aligned}\tag{3.33}$$

for some $0<t<1$. Here we used (2.7) in Lemma 2.1 for the last inequality. Thus we have

$$\begin{aligned}\mu_{\alpha,K}(\xi(\Lambda)\geqslant 2) &\leqslant \frac{1+2|\alpha|}{2}\max(||K_\Lambda||_1, ||J_\alpha[\Lambda]||_1)^2 \\ &\leqslant (\text{const.})||K_\Lambda||_1^2 = (\text{const.})\lambda_1(\Lambda)^2,\end{aligned} \tag{3.34}$$

where the constant depends only on K_Λ and α. Hence we obtain (3.32). $\square$

Remark 3.7. The above proof remains valid for a Polish space R if we replace compact sets Λ by measurable sets Λ with $\lambda(\Lambda)<\infty$ and functions with compact support by functions with $\lambda(\operatorname{supp} f)<\infty$.

Remark 3.8. Behind formula (3.29) there lies the relation

$$J_\alpha[\Lambda_0] = J_\alpha[\Lambda]_{\Lambda_0} + \alpha J_\alpha[\Lambda]_{\Lambda_0\Lambda_1}(I-\alpha J_\alpha[\Lambda]_{\Lambda_1})^{-1}J_\alpha[\Lambda]_{\Lambda_1\Lambda_0}, \tag{3.35}$$

where $\Lambda=\Lambda_0\cup\Lambda_1$ and $\Lambda_0\cap\Lambda_1=\emptyset$. It can be proved directly as follows. If T is a positive definite operator with bounded inverse T^{-1} on a Hilbert space H, then

$$(T^{-1})_{11} = (T_{11}-T_{12}(T_{22})^{-1}T_{21})^{-1} \tag{3.36}$$

whenever $T_{ij}=P_iTP_j$ $(i,j=1,2)$ for some orthogonal projection P_1 on H and $P_2=I-P_1$. Consequently, we have

$$\begin{aligned}\alpha J_\alpha[\Lambda_0] &= I-[I+\alpha K_{\Lambda_0}]^{-1} \\ &= I-[(I+\alpha K_\Lambda)_{\Lambda_0}]^{-1} \\ &= I-[\{(I-\alpha J_\alpha[\Lambda])^{-1}\}_{\Lambda_0}]^{-1} \\ &= I-\{(I-\alpha J_\alpha[\Lambda])_{\Lambda_0}-(I-\alpha J_\alpha[\Lambda])_{\Lambda_0\Lambda_1}(I-\alpha J_\alpha[\Lambda]_{\Lambda_1})^{-1}(I-\alpha J_\alpha[\Lambda])_{\Lambda_1\Lambda_0}\} \\ &= \alpha J_\alpha[\Lambda]_{\Lambda_0}+\alpha^2 J_\alpha[\Lambda]_{\Lambda_0\Lambda_1}(I-\alpha J_\alpha[\Lambda]_{\Lambda_1})^{-1}J_\alpha[\Lambda]_{\Lambda_1\Lambda_0}.\end{aligned} \tag{3.37}$$

Here we note that (3.36) implies

$$(T^{-1})_{11}\geqslant (T_{11})^{-1}. \tag{3.38}$$

This inequality will be used in the proof of Theorem 6.17.

3.3. An expansion formula and two convergence theorems

As a direct consequence of the expansion formula (2.10) of the Fredholm determinant in Lemma 2.1 we obtain the following "cumulant expansion" (cf. [36,38]).

Proposition 3.9. *Let f be a nonnegative measurable function with compact support Λ. Suppose $||\alpha K_\Lambda|| < 1$. Then we have*

$$\begin{aligned}
&-\log \int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) \\
&= \sum_{p=1}^{\infty}\sum_{n=1}^{p}(-1)^{p-1}\alpha^{n-1}\sum_{\substack{p_1,\dots,p_n\geqslant 1\\ p_1+\cdots+p_n=p}}\frac{1}{n\cdot p_1!\cdots p_n!}\int_{R^n}K(x_1,x_2)\cdots K(x_n,x_1) \\
&\quad\times\prod_{i=1}^{n} f(x_i)^{p_i}\lambda^{\otimes n}(dx_1\cdots dx_n) \\
&= \int_{R^1}K(x,x)f(x)\lambda(dx) - \frac{1}{2}\bigg(\int_{R^1}K(x,x)f(x)^2\lambda(dx) \\
&\quad + \alpha\int_{R^2}K(x,y)^2 f(x)f(y)\lambda^{\otimes 2}(dxdy)\bigg) + \dots . \qquad (3.39)
\end{aligned}$$

Proof. We can immediately obtain (3.39) by using Taylor expansion of the exponential function. □

One of the advantages of our definitions of those point processes is a convergence theorem.

Proposition 3.10. *Let $\{K^{(n)}\}_{n\geqslant 1}$ be integral operators with nonnegative definite continuous kernels $K^{(n)}(x,y)$. Assume that $K^{(n)}$ satisfies Condition* A (*or Condition* B) *and that $K^{(n)}(x,y)$ converges to a kernel $K(x,y)$ uniformly on each compact sets as n tends to infinity. Then the kernel $K(x,y)$ defines the integral operator K satisfying Condition* A (*or Condition* B, *respectively*). *Moreover, if $\alpha\in\{-1/m;\ m\in\mathbb{N}\}\cup\{2/m;\ m\in\mathbb{N}\}$, the measure $\mu_{\alpha,K^{(n)}}$ on Q associated with $K^{(n)}$ converges weakly to the measure $\mu_{\alpha,K}$ associated with K as n tends to infinity.*

From the nonnegative definiteness of operators we also obtain the following convergence theorem in terms of quadratic forms, which is sometimes much more useful than Proposition 3.10.

Proposition 3.11. *Let T_n, $n\geqslant 1$, be nonnegative definite trace class operators on a Hilbert space H. Assume that there exists a trace class operator T such that the quadratic form $\langle T_n f, f\rangle$ is monotone nondecreasing in n and converges to $\langle Tf, f\rangle$ as* *n goes to infinity for every* $f \in H$. *Then the Fredholm determinant* $\mathrm{Det}(I + T_n)$ *converges to* $\mathrm{Det}(I + T)$.

For the proofs we need the following fact whose proof can be found, e.g., in [34].

Lemma 3.12. *Let* T_n, T *be nonnegative definite symmetric operators on a Hilbert space. Suppose that as* $n \to \infty$, T_n *converges to* T *weakly and* $||T_n||_1$ *converges to* $||T||_1$. *Then* $||T_n - T||_1 \to 0$.

Proof of Proposition 3.10. First we prove the case of Condition A. Since the kernels $K^{(n)}(x, y)$ are continuous and nonnegative definite, the trace coincides with the integral on diagonal:

$$\mathrm{Tr}(K_\Lambda^{(n)}) = \int_\Lambda K^{(n)}(x, x)\lambda(dx) \tag{3.40}$$

for each compact sets Λ. Hence, the compact-uniform limit $K(x, y)$ is also continuous and nonnegative definite and

$$\mathrm{Tr}(K_\Lambda) = \int_\Lambda K(x, x)\lambda(dx) < \infty. \tag{3.41}$$

Moreover, we obtain $||K|| \leqslant C$ from $||K^{(n)}|| \leqslant C$ for all $n \in \mathbb{N}$. Thus, the operator K satisfies Condition A if $K^{(n)}$ satisfies Condition A.

Let $\varepsilon(n) = \sup_{(x,y) \in \Lambda \times \Lambda} |K(x, y) - K^{(n)}(x, y)|$. One can check that

$$||K_\Lambda - K_\Lambda^{(n)}|| \leqslant \varepsilon(n)\lambda(\Lambda) \tag{3.42}$$

and so since $K_\Lambda^{(n)}(x, y) \to K_\Lambda(x, y)$ is a compact uniform convergence then $K_\Lambda^{(n)}$ converges to K_Λ in uniform operator topology. Moreover,

$$|\mathrm{Tr}(K_\Lambda^{(n)}) - \mathrm{Tr}(K_\Lambda)| \leqslant \varepsilon(n)\lambda(\Lambda) \tag{3.43}$$

and so $||K_\Lambda^{(n)}||_1 \to ||K_\Lambda||_1$ as $n \to \infty$. Then by Lemma 3.12 we obtain

$$||K_\Lambda - K_\Lambda^{(n)}||_1 \to 0. \tag{3.44}$$

Since $\mathrm{Det}(I + \alpha K_\varphi)$ is continuous in K with respect to the norm $|| \cdot ||_1$ for each φ with compact support, one can conclude that the Laplace transform $\int e^{-\langle \xi, f \rangle} \mu_{\alpha, K^{(n)}}(d\xi)$ converges pointwise to $\int e^{-\langle \xi, f \rangle} \mu_{\alpha, K}(d\xi)$. Consequently, the probability measure $\mu_{\alpha, K^{(n)}}$ on the space $Q(R)$ converges weakly to $\mu_{\alpha, K}$. $\square$

It is also easy to prove the case of Condition B.

Proof of Proposition 3.11. Since $\langle T_n f, f\rangle$ converges to $\langle Tf, f\rangle$ for any $f \in H$, T_n converges T weakly. Since, in addition, the convergence is monotone nondecreasing and each $\langle T_n e_i, e_i\rangle$ is nonnegative, one obtains

$$||T_n||_1 = \sum_{i=1}^{\infty} \langle T_n e_i, e_i\rangle \nearrow \sum_{i=1}^{\infty} \langle Te_i, e_i\rangle = ||T||_1, \tag{3.45}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H. Thus, by Lemma 3.12, T_n converges to T in the norm $||\cdot||_1$ and hence $\mathrm{Det}(I+T_n) \to \mathrm{Det}(I+T)$ as $n \to \infty$. $\square$

4. Correlation functions

4.1. Definitions of correlation measures and correlation functions

Let us recall the definitions of correlation measures and correlation functions. Let μ be a probability measure on Q. Assume that (Q, μ) has no multiple points. For $\xi \in Q$ and any bounded measurable function f_n on R^n with compact support, denote

$$\langle \xi_n, f_n\rangle = \sum_{x_1, x_2, \dots, x_n \in \xi}^{*} f_n(x_1, x_2, \dots, x_n), \tag{4.1}$$

where $\sum^*$ denotes the sum over all mutually distinct points $x_1, x_2, \dots, x_n$. Then, for any function f with compact support one obtains

$$\exp(-\langle \xi, f\rangle) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \xi_n, \varphi_n\rangle, \tag{4.2}$$

where $\varphi_n(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} \varphi(x_i) = \prod_{i=1}^{n} (1 - \exp(-f(x_i)))$. In fact, the right hand side is a finite sum since $N = \xi(\mathrm{supp} f) < \infty$ and so (4.2) is easily obtained from the identity

$$\prod_{i=1}^{N} (1 - a_i) = \sum_{n=0}^{N} (-1)^n \sum_{\substack{I \subset \{1, \dots, N\} \\ |I| = n}} \prod_{i \in I} a_i. \tag{4.3}$$

If $\xi(\Lambda)^n$ is μ-integrable for each compact subset Λ of R, then $\langle \xi_n, f_n\rangle$ is μ-integrable for each bounded measurable function f_n with compact support on R^n and the formula

$$\int_Q \langle \xi_n, f_n\rangle \mu(d\xi) = \int_{R^n} f_n(x_1, \dots, x_n) \lambda_n(dx_1 \cdots dx_n) \tag{4.4}$$

defines a Radon measure λ_n on R^n which is called the nth correlation measure of μ. In particular, λ_1 is often called the intensity or the mean of μ.

Moreover, if $\int_Q \xi(\Lambda)^n \mu(d\xi), n \geqslant 1$, satisfy a suitable growth condition for each Λ so that if $\sum_{n=1}^{\infty}(1/n!)\int_Q \xi(\Lambda)^n \mu(d\xi) < \infty$ for each Λ, then one can integrate (4.2) and obtains the following expansion formula of the Laplace transform by correlation measures:

$$\int_Q \exp(-\langle \xi, f \rangle)\mu(d\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{R^n} \varphi_n(x_1, \dots, x_n)\lambda_n(dx_1 \cdots dx_n), \tag{4.5}$$

where f is a nonnegative measurable function with compact support.

Now let λ_1 be the intensity of μ. Fix a Radon measure λ on R and assume that λ_1 is absolutely continuous with respect to λ. Then the nth correlation measures λ_n of μ are absolutely continuous with respect to the direct product measures $\lambda^{\otimes n}$ whenever it exists. The Radon–Nikodym density $\rho_n(x_1, \dots, x_n)$ is called the n-th correlation function of μ (with respect to λ).

Moreover, if μ admits all the correlation functions and (4.5) holds, then one obtains the following expansion formula of the Laplace transform by correlation functions:

$$\begin{aligned} \int_Q \exp(-\langle \xi, f \rangle)\mu(d\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{R^n} & \varphi_n(x_1, \dots, x_n) \\ & \times \rho_n(x_1, \dots, x_n)\lambda^{\otimes n}(dx_1 \cdots dx_n). \end{aligned} \tag{4.6}$$

For instance, if μ is the Poisson point process with intensity λ, then $\lambda_n = \lambda^{\otimes n}$ and $\rho_n = 1$ for each $n \geqslant 1$ and (4.6) is the expansion of the exponential function.

The nth correlation function $\rho_n(x_1, \dots, x_n)$ is obviously symmetric in $x_1, \dots, x_n$ and so it is often convenient to write it as $\rho(X)$ where $X = \{x_1, \dots, x_n\}$.

Under this notation, the correlation functions $\rho(X)$ of the convolution $\mu = \mu^{(1)} * \mu^{(2)}$ are given by the formula

$$\rho(X) = \sum_{X = X_1 \sqcup X_2} \rho^{(1)}(X_1)\rho^{(2)}(X_2), \tag{4.7}$$

where $\sum_{X = X_1 \sqcup X_2}$ stands for the summation over all disjoint subsets X_1, X_2 of X with $X_1 \sqcup X_2 = X$ and $\rho^{(i)}$ is the correlation function of $\mu^{(i)}$, $i = 1, 2$. Formally, (4.7) follows from

$$\int_Q \mu(d\xi)e^{-\langle \xi, f \rangle} = \int_Q \mu^{(1)}(d\xi)e^{-\langle \xi, f \rangle} \cdot \int_Q \mu^{(2)}(d\xi)e^{-\langle \xi, f \rangle} \tag{4.8}$$

by using (4.6).

4.2. α-Determinants and correlation functions

Now we proceed to prove the last part of Theorem 1.2 assuming the other parts.

Theorem 4.1. *Assume Condition* A *and* $\alpha \in \{2/m;\ m \in \mathbb{N}\} \cup \{-1/m;\ m \in \mathbb{N}\}$. *Then all the correlation functions* $\rho_{n,\alpha,K}$ *of the probability measure* $\mu_{\alpha,K}$ *defined in Theorem* 1.2 *exist and are given by the formula*

$$\rho_{n,\alpha,K}(x_1, \dots, x_n) = \det_\alpha (K(x_i, x_j))_{i,j=1}^n \quad (n \geqslant 1) \tag{4.9}$$

and

$$\rho_{0,\alpha,K} = 1. \tag{4.10}$$

Moreover, if we assume, in addition, $||\varphi||_\infty ||\alpha K|| < 1$ *when* $\alpha > 0$, *we have the expansion formula*

$$\int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle)$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{\Lambda^n} \det_\alpha (K(x_i, x_j))_{i,j=1}^n \varphi_n(x_1, \dots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n). \tag{4.11}$$

In order to prove the existence of correlation functions, we need the following estimates of the probabilities of basic events for $\mu_{\alpha,K}$.

Up to now we have discussed things only for $\alpha = \pm 1/m (m \in \mathbb{N})$. But the proofs below work also in the case $\alpha = 2/m (m \in \mathbb{N})$ which will be discussed in Section 6.

Lemma 4.2. *For any compact set* $\Lambda \subset R$, *the following estimates hold*:

$$\mu_{\alpha,K}(\xi(\Lambda) = k) \leqslant \frac{1}{k!} \left(\frac{||K_\Lambda||_1}{1 - ||\alpha K_\Lambda||} \right)^k \quad \text{if } \alpha < 0, \tag{4.12}$$

$$\mu_{\alpha,K}(\xi(\Lambda) = k) \leqslant C_{\Lambda,\gamma} \left(\frac{\gamma ||\alpha K_\Lambda||}{1 + ||\alpha K_\Lambda||} \right)^k \quad \text{if } \alpha > 0 \tag{4.13}$$

for any $\gamma > 1$ *and some* $C_{\Lambda,\gamma} > 0$.

Proof. In the case of $\alpha = -1$ it is immediate from Lemma 2.1(i) and (ii) since

$$\mu_{-1,K}(\xi(\Lambda) = k) = \mathrm{Det}(I - K_\Lambda)\, \mathrm{Tr}(\wedge^k J_{-1}[\Lambda]) \tag{4.14}$$

for any $k \in \mathbb{N}$. In the case of $\alpha = -1/m\ (m \in \mathbb{N})$, $\mu_{-1/m,K}$ is the m-fold convolution of $\mu_{-1,K/m}$ and so

$$\mu_{-1/m,K}(\xi(\Lambda) = k) = \sum_{j_1 + \cdots + j_m = k} \prod_{i=1}^m \mu_{-1,K/m}(\xi(\Lambda) = j_i)$$

$$\leqslant \sum_{j_1+\cdots+j_m=k} \frac{1}{j_1!\cdots j_m!}\left(\frac{||\alpha K_\Lambda||_1}{1-||\alpha K_\Lambda||}\right)^{j_1+\cdots+j_m}$$
$$=\frac{1}{k!}\left(\frac{||K_\Lambda||_1}{1-||\alpha K_\Lambda||}\right)^k. \tag{4.15}$$

In the case of $\alpha > 0$, note that

$$\sum_{k=0}^{\infty} z^k \mu_{\alpha,K}(\xi(\Lambda)=k) = \mathrm{Det}(I+\alpha K_\Lambda)^{-1/\alpha}\,\mathrm{Det}(I - z\alpha J_\alpha[\Lambda])^{-1/\alpha} \tag{4.16}$$

and then the right-hand side is analytic in z whenever $|z|\cdot||\alpha J_\alpha[\Lambda]|| < 1$. Thus, there exists a constant $C_{\Lambda,\gamma} > 0$ for $\gamma > 1$ so that

$$\mu_{\alpha,K}(\xi(\Lambda)=k) \leqslant C_{\Lambda,\gamma}(\gamma||\alpha J_\alpha[\Lambda]||)^k$$
$$\leqslant C_{\Lambda,\gamma}\left(\frac{\gamma||\alpha K_\Lambda||}{1+||\alpha K_\Lambda||}\right)^k. \qquad \square \tag{4.17}$$

Proof of Theorem 4.1. First assume $\alpha > 0$. By Lemma 4.2, for each compact set Λ of R, we have

$$\mu_{\alpha,K}(\xi(\Lambda)=k) \leqslant C_{\Lambda,\gamma}\beta^k, \tag{4.18}$$

where $\beta = \gamma||\alpha K_\Lambda||(1+||\alpha K_\Lambda||)^{-1}$ for any $\gamma > 1$. Since $||\varphi||_\infty ||\alpha K_\Lambda|| < 1$, we get $||\varphi||_\infty \beta(1-\beta)^{-1} < 1$ if we take γ sufficiently near to 1. Consequently,

$$\left|\sum_{n\geqslant N} \frac{(-1)^n}{n!}\int_Q \langle \xi_n, \varphi_n \rangle \mu_{\alpha,K}(d\xi)\right|$$
$$\leqslant \sum_{n\geqslant N} \frac{||\varphi||_\infty^n}{n!}\int_Q \xi(\Lambda)(\xi(\Lambda)-1)\cdots(\xi(\Lambda)-n+1)\mu_{\alpha,K}(d\xi)$$
$$\leqslant \sum_{n\geqslant N} \frac{||\varphi||_\infty^n}{n!}\frac{C_{\Lambda,\gamma} n!\beta^n}{(1-\beta)^{n+1}} < \infty. \tag{4.19}$$

Hence for any bounded measurable function with compact support the formula (4.4) is well defined and the correlation measure $\lambda_{n,\alpha,K}$ of $\mu_{\alpha,K}$ exist for each n. Thanks to estimate (4.19), we can integrate the both hand side (4.2) safely to obtain

$$\int_{Q(\Lambda)} \exp(-\langle \xi, f\rangle)\mu_{\Lambda,\alpha,K}(d\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\int_{\Lambda^n} \varphi_n(x_1,\ldots,x_n)\lambda_{n,\alpha,K}(dx_1\ldots dx_n). \tag{4.20}$$

On the other hand, by expansion (2.16) of α-determinant in Theorem 2.4, we obtain

$$\int_{Q(\Lambda)} \exp(-\langle \xi, f \rangle)\mu_{\Lambda,\alpha,K}(d\xi)$$

$$= \mathrm{Det}(I + \alpha K_\varphi)^{-1/\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det{}_\alpha(K_\varphi(x_i, x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1 \cdots dx_n)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det{}_\alpha(K(x_i, x_j))_{i,j=1}^n \varphi_n(x_1, \ldots, x_n)\lambda^{\otimes n}(dx_1 \cdots dx_n). \tag{4.21}$$

Comparing (4.21) with (4.20), we can conclude that $\lambda_{n,\alpha,K}$ is absolutely continuous with respect to $\lambda^{\otimes n}$ and

$$\rho_{n,\alpha,K}(x_1, x_2, \ldots, x_n) = \det{}_\alpha(K(x_i, x_j))_{i,j=1}^n. \tag{4.22}$$

In the case where $\alpha < 0$, a similar argument shows (4.22). $\square$

4.3. Correlation inequalities

Finally, we had better to notice that fermion and fermion-like ($\alpha = -1/m < 0$) point processes have "repulsive" character and boson and boson-like ($\alpha = 1/m > 0$) point processes have "attractive" character.

Proposition 4.3. *The correlation functions of the probability measure $\mu_{\alpha,K}$ satisfy the following inequalities*:

$$\rho_{n,\alpha,K}(x_1, \ldots, x_n) \geqslant \rho_{1,\alpha,K}(x_1) \ldots \rho_{1,\alpha,K}(x_n) \quad \text{if } \alpha = 1/m > 0 \tag{4.23}$$

and

$$\rho_{n,\alpha,K}(x_1, \ldots, x_n) \leqslant \rho_{1,\alpha,K}(x_1) \ldots \rho_{1,\alpha,K}(x_n) \quad \text{if } \alpha = -1/m < 0, \tag{4.24}$$

where m is a positive integer.

Furthermore, if $\alpha = -1$,

$$\rho_{n+m+\ell,-1,K}(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_\ell)\rho_{\ell,-1,K}(z_1, \ldots, z_\ell)$$
$$\leqslant \rho_{n+\ell,-1,K}(x_1, \ldots, x_n, z_1, \ldots, z_\ell)\rho_{m+\ell,-1,K}(y_1, \ldots, y_m, z_1, \ldots, z_\ell). \tag{4.25}$$

Proof. It suffices to prove the assertions only for $\alpha = \pm 1$. Indeed, $\mu_{\alpha/m,K}$ is the m-fold convolution of $\mu_{\alpha,K/m}$ and it follows from (4.7) that

$$\rho_{n,\alpha/m,K}(X) = \sum_{X_1 \sqcup X_2 \sqcup \cdots \sqcup X_m = X} \prod_{j=1}^{m} \rho_{n,\alpha,K/m}(X_j)$$

$$\left\{ \begin{matrix} \geqslant \\ \leqslant \end{matrix} \right\} \sum_{X_1 \sqcup X_2 \sqcup \cdots \sqcup X_m = X} \prod_{i=1}^{n} \rho_{1,\alpha,K/m}(x_i)$$

$$= \prod_{i=1}^{n} \rho_{1,\alpha,K}(x_i) = \prod_{i=1}^{n} \rho_{1,\alpha/m,K}(x_i), \tag{4.26}$$

where $X = \{x_1, \ldots, x_n\}$ and the summation is taken over all mutually disjoint subsets $X_1, \ldots, X_m$ of X with $X_1 \cup \cdots \cup X_m = X$. Here we used the fact that $\rho_{1,\alpha,K}$ depends only on K.

First we consider the case $\alpha = -1$. Inequalities (4.24) and (4.25) follow from an inequality for a nonnegative definite, 3 by 3 block matrix:

$$\det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \det A_{22} \leqslant \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \det \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}. \tag{4.27}$$

For the case $\alpha = 1$, we immediately obtain (4.23) by considering the highest coefficient of $\mathrm{per}(A(t))$ and putting $t = 1$ in the following Theorem 4.4 obtained by Lieb [19]. $\square$

Theorem 4.4. *Let*

$$A(t) = \begin{pmatrix} tB & C \\ C^* & D \end{pmatrix}, \tag{4.28}$$

where t is an indeterminate over $\mathbb{C}$. Assume that $A(1)$ is nonnegative definite. Then all the coefficients of the polynomial per $A(t)$ *are real and nonnegative.*

5. Limit theorems

5.1. Convolution kernels

In this section we restrict ourselves to convolution operators on $\mathbb{R}^d$ and qdiscuss basic limit theorems, namely, the law of large numbers, the central limit theorem and a large deviation result. Throughout this section, we assume $\alpha \in \{\pm 1/m;\ m \in \mathbb{N}\} \cup \{2/m;\ m \in \mathbb{N}\}$. We continue to assume Condition

A in Theorem 1.2, which can be restated in terms of the Fourier transform as follows.

Lemma 5.1. *Assume that K is a convolution operator on $L^2(\mathbb{R}^d)$ with continuous kernel k. Then the following two statements are mutually equivalent*:

(a) *K satisfies Condition* A.

(b) *The convolution kernel k is the Fourier transform of an even function $\hat{k}$ in $L^1(\mathbb{R}^d)$*

$$k(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{k}(t) e^{-ix\cdot t}\, dt \tag{5.1}$$

and $\hat{k}$ takes values in $[0, \infty)$ if $\alpha > 0$ and in $[0, |\alpha|^{-1}]$ if $\alpha < 0$.

Remark 5.2. Note that (b) implies $k \in L^2(\mathbb{R}^d)$. But k does not necessarily belong to $L^1(\mathbb{R}^d)$. Indeed, the sine kernel $\sin \pi x/\pi x$ is a typical example which satisfies Condition A but does not belong to $L^1(\mathbb{R}^1)$.

Remark 5.3. For the convolution operator K there exist no localized eigenfunctions. Indeed, if there existed an eigenfunction f with compact support, say Λ, associated with an eigenvalue α, then its translations $f_x = f(\cdot + x)$ would be also eigenfunctions. Thus, α would be an eigenvalue of $K_{\tilde{\Lambda}}$ with infinite multiplicity whenever a compact set $\tilde{\Lambda}$ contains an open neighborhood of Λ. This would contradict the compactness of operators $K_{\tilde{\Lambda}}$.

On the other hand, the convolution operator K itself may have an eigenvalue with infinite multiplicity. For instance, consider the sine kernel $k(x) = \sin \pi x/\pi x$ on $\mathbb{R}^1$. Then the function $k_x(y) = k(x - y)$ is an eigenfunction with eigenvalue 1 for each $x \in \mathbb{R}^1$.

From now on we always assume the kernel $k(x)$ satisfy the conditions given in above Lemma 5.1. Thus, it follows from Theorem 4.1

$$\rho_{1,\alpha,K}(x) = k(0), \quad \rho_{2,\alpha,K}(x, y) = k(0)^2 + \alpha|k(x - y)|^2, \tag{5.2}$$

etc.

5.2. *Law of large numbers*

First we compute the limiting covariance in generic case.

Lemma 5.4. *Let f be a bounded measurable function on $\mathbb{R}^d$ with compact support and set $f_N = f(\cdot/N)$. Then, as $N \to \infty$,*

$$\int_Q \mu_{\alpha,K}(d\xi)\left(\langle \xi, f_N\rangle - \int_Q \langle \xi, f_N\rangle \mu_{\alpha,K}(d\xi)\right)^2$$
$$= \int_Q \langle \xi, f_N\rangle^2 \mu_{\alpha,K}(d\xi) - \left(\int_Q \langle \xi, f_N\rangle \mu_{\alpha,K}(d\xi)\right)^2$$
$$\sim N^d \int_{\mathbb{R}^d} f(x)^2\, dx \times \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{k}(t)(1+\alpha\hat{k}(t))\, dt. \tag{5.3}$$

Proof. By the definition of the correlation function

$$\int_Q \langle \xi, f_N\rangle \mu_{\alpha,K}(d\xi) = \int_{\mathbb{R}^d} f_N(x)\rho_{1,\alpha,K}(x)\, dx,$$

$$\int_Q \langle \xi, f_N\rangle^2 \mu_{\alpha,K}(d\xi) = \int_{\mathbb{R}^d\times\mathbb{R}^d} f_N(x) f_N(y)\rho_{2,\alpha,K}(x,y)\, dx\, dy$$
$$+ \int_{\mathbb{R}^d} f_N(x)^2 \rho_{1,\alpha,K}(x)\, dx. \tag{5.4}$$

Hence from (5.2) we can compute the left-hand side of (5.4) directly to obtain

$$\text{(LHS)} = \int_{\mathbb{R}^d\times\mathbb{R}^d} f_N(x) f_N(y)(k(0)^2 + \alpha|k(x-y)|^2)\, dx\, dy + \int_{\mathbb{R}^d} f_N(x)^2 k(0)\, dx$$
$$- \left(\int_{\mathbb{R}^d} f_N(x) k(0)\, dx\right)^2$$
$$= \int_{\mathbb{R}^d} f_N(x)^2 k(0)\, dx + \alpha \int_{\mathbb{R}^d\times\mathbb{R}^d} f_N(x) f_N(y)|k(x-y)|^2\, dx\, dy$$
$$= N^d\left(\int_{\mathbb{R}^d} f(x)^2 k(0)\, dx + \alpha \int_{\mathbb{R}^d} k(u)^2 du \int_{\mathbb{R}^d} f(x) f\left(x + \frac{u}{N}\right) dx\right)$$
$$\sim N^d\left(k(0) + \alpha \int_{\mathbb{R}^d} |k(u)|^2\, du\right)\int_{\mathbb{R}^d} f(x)^2\, dx$$
$$= N^d\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{k}(t)(1+\alpha\hat{k}(t))\, dt \times \int_{\mathbb{R}^d} f(x)^2\, dx. \qquad \square \tag{5.5}$$

Now let us state our law of large numbers. It is an immediate consequence of Lemma 5.4.

Proposition 5.5. *Let f be a bounded measurable function on $\mathbb{R}^d$ with compact support. Then*

$$\left\langle \xi, \frac{f_N}{N^d} \right\rangle \to \int_{\mathbb{R}^d} f(x)k(0)\,dx \quad \mu_{\alpha,K}\text{-a.e. } \xi \text{ and in } L^1(Q, \mu_{\alpha,K}), \tag{5.6}$$

where $f_N(\cdot) = f(\cdot/N)$.

Remark 5.6. If K is a convolution operator on $\mathbb{R}^d$, the translation turns out to be mixing under $\mu_{\alpha,K}$. Further properties are known when $\alpha = -1$. The totally mixing property (the mixing property with arbitrary multiplicity) and the absolute continuity of the spectrum are proved in [37]. Ergodic properties for the case where $R = \mathbb{Z}^d$ such as entropy, Gibbs property and Bernoullicity are treated in [22,31].

5.3. Central limit theorem

The central limit theorem for the point processes for $\alpha = -1$ was discussed in (cf. [35,37,38]). See also Remark 5.8. For general α, it can be shown by using cumulant expansion thanks to the fact that the Laplace transform is given as the Fredholm determinant to the power $-1/\alpha$.

Proposition 5.7. *Let f be a bounded measurable function on $\mathbb{R}^d$ with compact support and assume $\int_{\mathbb{R}^d} f(x)\,dx = 0$. Then*

$$\lim_{N\to\infty} \int_Q \mu_{\alpha,K}(d\xi) \exp\left(i\left\langle \xi, \frac{f_N}{N^{d/2}} \right\rangle \right) = \exp\left(-\frac{1}{2}\sigma^2_{\alpha,K} \|f\|_2^2 \right), \tag{5.7}$$

where

$$\sigma^2_{\alpha,K} = k(0) + \alpha \int_{\mathbb{R}^d} |k(x)|^2\,dx \tag{5.8}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{k}(t)(1 + \alpha\hat{k}(t))\,dt \tag{5.9}$$

and $f_N(\cdot) = f(\cdot/N)$.

Proof. Set $\Lambda = \operatorname{supp} f$. Let $\varphi_N(x) = 1 - \exp(if_N(x)/N^{d/2})$, $N\Lambda = \{Nx;\ x \in \Lambda\}$, and $L_N = \varphi_N K_{N\Lambda}$. Since $\int_{\mathbb{R}^d} f(x)\,dx = 0$, we get

$$\mathrm{Tr}(L_N) = \int_{\mathbb{R}^d} \left(1 - \exp\left(\frac{i}{N^{d/2}} f\left(\frac{x}{N}\right) \right) + \frac{i}{N^{d/2}} f\left(\frac{x}{N}\right) \right) k(0)\,dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^d} f(x)^2 k(0)\,dx + O\left(\frac{1}{N^{d/2}}\right) \\
&\to \frac{1}{2}\int_{\mathbb{R}^d} f(x)^2 k(0)\,dx. \qquad (5.10)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{Tr}(L_N^2) &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} k(x)k(-x)f(y)f\left(y+\frac{x}{N}\right)dx\,dy + O\left(\frac{1}{N^{d/2}}\right) \\
&\to -\int_{\mathbb{R}^d} |k(x)|^2\,dx \int_{\mathbb{R}^d} f(x)^2\,dx. \qquad (5.11)
\end{aligned}
$$

Using (2.6) in Lemma 2.1(i) and Theorem 1.5, we obtain the following estimates: for sufficiently large N,

$$
\begin{aligned}
&\left| -\log \int_Q \mu_{\alpha,K}(d\xi)\exp\left(i\left\langle \xi, \frac{f_N}{N^{d/2}}\right\rangle\right) - \mathrm{Tr}(L_N) + \frac{\alpha}{2}\mathrm{Tr}(L_N^2)\right| \\
&\leqslant \sum_{n\geqslant 3} \frac{|\alpha|^{n-1}}{n}\mathrm{Tr}(|L_N|^n) \\
&\leqslant \sum_{n\geqslant 3} \frac{|\alpha|^{n-1}}{n}||L_N||^{n-2}\,\mathrm{Tr}(|L_N|^2) \\
&\leqslant -|\alpha|\log(1-||\alpha K||\cdot||\varphi_N||_\infty)\,\mathrm{Tr}(|L_N|^2) \qquad (5.12)
\end{aligned}
$$

and since $K_{N\Lambda}$ is bounded and nonnegative definite

$$
\begin{aligned}
\mathrm{Tr}(|L_N|^2) &\leqslant ||\varphi_N||_\infty^2\,\mathrm{Tr}(K_{N\Lambda}^2) \\
&\leqslant ||\varphi_N||_\infty^2\,||K_{N\Lambda}||\,\mathrm{Tr}(K_{N\Lambda}) \\
&\leqslant \left(\frac{||f||_\infty}{N^{d/2}}\right)^2 ||K||N^d|\Lambda|k(0) \\
&= k(0)||f||_\infty^2\,||K||\,|\Lambda|. \qquad (5.13)
\end{aligned}
$$

Here we used $||\varphi_N||_\infty \leqslant ||f||_\infty / N^{d/2}$. From (5.12) and (5.13) it follows

$$
\begin{aligned}
&- \lim_{N\to\infty} \log \int_Q \mu_{\alpha,K}(d\xi) \exp\left(i \left\langle \xi, \frac{f_N}{N^{d/2}} \right\rangle \right) \\
&= \frac{1}{2}\left(k(0) + \alpha \int_{\mathbb{R}^d} |k(x)|^2\, dx \right) \int_{\mathbb{R}^d} f(x)^2\, dx \\
&= \frac{1}{2}\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{k}(t)(1+\alpha\hat{k}(t))\, dt \int_{\mathbb{R}^d} f(x)^2\, dx. \qquad (5.14)
\end{aligned}
$$

Remark 5.8. In the case where $\alpha<0$ the range of the Fourier transform $\hat{k}$ is crucial for the asymptotic behavior of the variance. For instance, if $\alpha = -1$ and $\hat{k}$ takes only two values, 0 and 1, then the quantity $\sigma^2_{-1,K}$ vanishes and the standard scaling factor $N^{-d/2}$ loses its meaning. The sine kernel $k(x) = \sin \pi x/\pi x$ is a typical case among such degenerated cases. Indeed, if we denote the fermion process associated with it by $\mu_{-1,\mathrm{sine}}$, then one obtains the following $\log N$ behavior:

$$
\begin{aligned}
&\int_Q \langle \xi, f_N \rangle^2 \mu_{-1,\mathrm{sine}}(d\xi) - \left(\int_Q \langle \xi, f_N \rangle \mu_{-1,\mathrm{sine}}(d\xi) \right)^2 \\
&\sim (\log N) \frac{1}{2\pi^2} \sum (f(x+0) - f(x-0))^2 \qquad (5.15)
\end{aligned}
$$

for functions f of bounded variation and with compact support provided that its jumps are square summable. In particular, if we take the indicator function of unit interval $[0,1]$ as f,

$$
\int_Q \langle \xi, f_N \rangle^2 \mu_{-1,\mathrm{sine}}(d\xi) - \left(\int_Q \langle \xi, f_N \rangle \mu_{-1,\mathrm{sine}}(d\xi) \right)^2 = \frac{1}{\pi^2} \log N + O(1). \qquad (5.16)
$$

This comes from the well-known $\log N$ behavior for Dirichlet kernel in Fourier analysis (cf. [42]). In fact,

$$
\begin{aligned}
&\int_Q \langle \xi, f_N \rangle^2 \mu_{-1,\mathrm{sine}}(d\xi) - \left(\int_Q \langle \xi, f_N \rangle \mu_{-1,\mathrm{sine}}(d\xi) \right)^2 \\
&= \int_0^N dx - \int_0^N \int_0^N \left(\frac{\sin \pi(x-y)}{\pi(x-y)} \right)^2 dx\, dy \\
&= N\left(1 - \int_{-N}^N \left(\frac{\sin \pi u}{\pi u} \right)^2 du \right) + \frac{1}{\pi^2} \int_0^N \frac{1-\cos 2\pi u}{u}\, du
\end{aligned}
$$

$$= O(1) + \left(\frac{1}{\pi^2} \int_1^N \frac{du}{u} + O(1) \right)$$
$$= \frac{1}{\pi^2} \log N + O(1). \tag{5.17}$$

The central limit theorem does hold for indicator functions of an interval under this $\log N$ scaling. It was first proved by Costin and Lebowitz [7]. Further discussions were given for general f by Soshnikov [35,38].

5.4. A large deviation result

Proposition 5.9. *Let f be a nonnegative measurable function on $\mathbb{R}^d$ with compact support and set $f_N(\cdot) = f(\cdot/N)$. Suppose, in addition, that $\|\alpha K\| \leqslant 1$ when $\alpha > 0$. Then*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f_N \rangle)$$
$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dx \Phi_\alpha(\hat{k}(t), f(x)), \tag{5.18}$$

where we set

$$\Phi_\alpha(\kappa, u) = -\frac{1}{\alpha} \log(1 + \alpha\kappa(1 - e^{-u})). \tag{5.19}$$

Remark 5.10. The quantity $\Phi_\alpha(\kappa, u)$ is the logarithm of the Laplace transform of a generalized binomial distribution:

$$\exp \Phi_\alpha(\kappa, u) = (1 + \alpha\kappa(1 - e^{-u}))^{-1/\alpha} = (1 + \alpha\kappa)^{-1/\alpha} \sum_{n=0}^{\infty} \frac{c^{(n)}(\alpha)}{n!} j_\alpha^n e^{-nu}, \tag{5.20}$$

where $c^{(n)}(\alpha) = \prod_{i=0}^{n-1} (1 + i\alpha)$ and $j_\alpha = \kappa(1 + \alpha\kappa)^{-1}$ as in the introduction.

Proof of Proposition 5.9. First we assume $k \in L^1$ in addition to $\hat{k} \in L^1$. Let $\varphi = 1 - \exp(-f)$, $\varphi_N = 1 - \exp(-f_N)$ and $\Lambda = \operatorname{supp} f$. Note that $K_{\varphi_N} = \sqrt{\varphi_N} K_{N\Lambda} \sqrt{\varphi_N}$, and by Lemma 2.1(i) we obtain

$$\operatorname{Tr}(K_{\varphi_N}^n) \leqslant \|\varphi_N\|_\infty^n \cdot \|K_{N\Lambda}\|^{n-1} \operatorname{Tr}(K_{N\Lambda})$$
$$= \|\varphi\|_\infty^n \|K\|^{n-1} k(0) N^d |\Lambda| \tag{5.21}$$

and

$$\begin{aligned}
\mathrm{Tr}(K^n_{\varphi_N}) &= \int_{\mathbb{R}^{dn}} k(x_1 - x_2)\cdots k(x_n - x_1)\varphi_N(x_1)\cdots\varphi_N(x_n)\, dx_1 \dots dx_n \\
&= \int_{\mathbb{R}^{dn}} k(y_1)\cdots k(y_{n-1})k(-y_1 - \cdots - y_{n-1}) \\
&\quad \times \varphi\Big(\frac{x_1}{N}\Big)\varphi\Big(\frac{x_1+y_1}{N}\Big)\cdots\varphi\Big(\frac{x_1+y_1+\cdots+y_{n-1}}{N}\Big)\, dx_1\, dy_1 \dots dy_{n-1} \\
&= \int_{\mathbb{R}^{dn}} k(y_1)\cdots k(y_{n-1})k(-y_1 - \cdots - y_{n-1}) \\
&\quad \times \varphi(x)\varphi\Big(x + \frac{y_1}{N}\Big)\cdots\varphi\Big(x + \frac{y_1+\cdots+y_{n-1}}{N}\Big)N^d\, dx\, dy_1 \dots dy_{n-1} \\
&\sim N^d \int_{\mathbb{R}^{d(n-1)}} dy_1 \dots dy_{n-1} k(y_1)\cdots k(y_{n-1})k(-y_1 - \cdots - y_{n-1}) \\
&\quad \times \int_{\mathbb{R}^d} dx \varphi(x)^n \\
&\sim N^d \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} dt \hat{k}(t)^n \int_{\mathbb{R}^d} \varphi(x)^n\, dx.
\end{aligned} \tag{5.22}$$

By the dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^d} \log \mathrm{Det}(I + \alpha K_{\varphi_N}) \\
&\quad = -\lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n} \frac{1}{N^d} \mathrm{Tr}(K^n_{\varphi_N}) \\
&\quad = -\sum_{n=1}^{\infty} \frac{(-\alpha)^n}{n} \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} dt \hat{k}(t)^n \int_{\mathbb{R}^d} \varphi(x)^n\, dx \\
&\quad = \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dx \log(1 + \alpha\hat{k}(t)\varphi(x)).
\end{aligned} \tag{5.23}$$

Now we consider the general case. If $\hat{k}$ is in L^1 then we can find a sequence $\{k_n\}$ such that both k_n and $\widehat{k_n}$ are in L^1 and $\|\hat{k} - \widehat{k_n}\|_{L^1} \to 0$ as $n \to \infty$. Consequently we obtain Proposition 5.9 from the following lemma. $\square$

Lemma 5.11. *Let f be a nonnegative measurable function of compact support and $\varphi(x) = 1 - e^{-f(x)}$. Suppose $\|\alpha K\| \leqslant 1$. Then,*

$$\int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} dx \log(1 + \alpha\hat{k}(t)\varphi(x)) \tag{5.24}$$

is Lipschitz continuous in $\hat{k}$ *with respect to the norm* $||\cdot||_{L^1}$ *and so are the quantities*

$$\frac{1}{N^d}\log \operatorname{Det}(1+\alpha K_{\varphi_N}) \tag{5.25}$$

uniformly in N *where* $\varphi_N = \varphi(\cdot/N)$.

Proof. We only give a proof to the second assertion because the first one is proved in a similar and easier way.

Let k_0, k_1 be such that $\hat{k}_0, \hat{k}_1 \in L^1$ and set $k_r = (1-r)k_0 + rk_1$ $(0 \leqslant r \leqslant 1)$. Denote by $K^{(r)}$ the operator corresponding to k_r. Then, by Lemma 2.1(iv),

$$\frac{d}{dr}\log \operatorname{Det}(1+\alpha K^{(r)}_{\varphi_N}) = \alpha \operatorname{Tr}\left((1+\alpha K^{(r)}_{\varphi_N})^{-1}\frac{d}{dr}K^{(r)}_{\varphi_N}\right) \tag{5.26}$$

and

$$\begin{aligned}\frac{d}{dr}K^{(r)}_{\varphi_N}(x,y) &= (\varphi_N(x)\varphi_N(y))^{1/2}(k_1-k_0)(x-y)\\ &= (\varphi_N(x)\varphi_N(y))^{1/2}\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} (\hat{k}_1-\hat{k}_0)(t)e^{-i(x-y)\cdot t}\,dt.\end{aligned} \tag{5.27}$$

$$\frac{d}{dr}\log \operatorname{Det}(1+\alpha K^{(r)}_{\varphi_N}) = \alpha\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \langle (1+\alpha K^{(r)}_{\varphi_N})^{-1}\psi^t_N, \psi^t_N\rangle (\hat{k}_1-\hat{k}_0)(t)\,dt, \tag{5.28}$$

where

$$\psi^t_N(x) = e^{-ix\cdot t}\varphi_N(x)^{1/2}. \tag{5.29}$$

Now noting $||\alpha K^{(r)}|| \leqslant 1$ and

$$\begin{aligned}\langle (1+\alpha K^{(r)}_{\varphi_N})^{-1}\psi^t_N, \psi^t_N\rangle &\leqslant (1-||\varphi||_\infty)^{-1}\langle \psi^t_N, \psi^t_N\rangle\\ &= N^d(1-||\varphi||_\infty)^{-1}||\varphi||_\infty |\operatorname{supp}\varphi|,\end{aligned} \tag{5.30}$$

we obtain

$$\left|\frac{d}{dr}\log \operatorname{Det}(1+\alpha K^{(r)}_{\varphi_N})\right| \leqslant C|\alpha|N^d||\hat{k}_1-\hat{k}_0||_{L^1} \tag{5.31}$$

with $C = (1-||\varphi||_\infty)^{-1}||\varphi||_\infty|\operatorname{supp}\varphi|/(2\pi)^d$.

Consequently,

$$\left|\frac{1}{N^d}\log \operatorname{Det}(1+\alpha K^{(1)}_{\varphi_N}) - \frac{1}{N^d}\log \operatorname{Det}(1+\alpha K^{(0)}_{\varphi_N})\right| \leqslant C|\alpha|\cdot||\hat{k}_1-\hat{k}_0||_{L^1}. \quad \square \tag{5.32}$$

If we consider the degenerated fermion and fermion-like point processes, we obtain the following, rather strange result from Proposition 5.9. One might say that a strong mean field theory works for degenerated fermion and fermion-like point processes.

Corollary 5.12. *Let f be a nonnegative measurable function on $\mathbb{R}^d$ with compact support and set $f_N(\cdot) = f(\cdot/N)$. Suppose $\alpha = -1/m$, $m \in \mathbb{N}$ and $\hat{k}$ takes only two values* 0 *and m. Then,*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \int_Q \mu_{-1/m,K}(d\xi) \exp(-\langle \xi, f_N \rangle) = -k(0) \int_{\mathbb{R}^d} f(x)\, dx. \tag{5.33}$$

6. Further properties

6.1. Palm measures

Throughout this section, we assume that λ is nonatomic and we deal only with point processes which have no multiple points and admit all the correlation measures λ_n.

Definition 6.1. Let μ be a probability measure on Q. If μ has mean λ_1, one can define a probability measure μ^x on Q for λ_1-a.e. x by the disintegration formula

$$\int_Q \mu(d\xi) \int_R \xi(dx) u(\xi, x) = \int_R \lambda_1(dx) \int_Q \mu^x(d\xi) u(\xi + \delta_x, x) \tag{6.1}$$

for any bounded measurable function $u(\xi, x)$ on $Q \times R$ with compact support in x. The probability measure μ^x on Q is called the Palm measure or Palm-Khintchin measure or sometimes Kendall measure of μ.

Remark 6.2. In our definition, the Palm measure μ^x of μ is supported on the set of ξ satisfying $\xi\{x\} = 0$, that is,

$$\mu^x(\xi \in Q;\ \xi\{x\} = 0) = 1. \tag{6.2}$$

For instance, if Π is the Poisson point process with intensity λ, then its Palm measure Π^x coincides with Π for λ-a.e. x. Indeed, differentiating

$$\int_Q \Pi(d\xi) e^{-\langle \xi, f + tg \rangle} = \exp\left(- \int_R (1 - e^{-f(x) - tg(x)}) \lambda(dx) \right) \tag{6.3}$$

in t at $t = 0$, one finds

$$\int_Q \Pi(d\xi) \langle \xi, g \rangle e^{-\langle \xi, f \rangle} = \int_R \lambda(dx) g(x) e^{-f(x)} \int_Q \Pi(d\xi) e^{-\langle \xi, f \rangle} \tag{6.4}$$

for any nonnegative measurable functions f and g with compact support. If we set $u(\xi, x) = g(x)e^{-\langle \xi, f \rangle}$, then those functions span the space $L^1(Q \times R, \Pi \otimes \lambda)$. Hence, (6.1) holds for $\mu = \Pi$ with $\mu^x = \Pi$.

Similarly, for $n \geqslant 2$ the nth Palm measure is defined as the probability measure $\mu^{x_1, \dots, x_n}$ on Q for λ_n-a.e. $(x_1, \dots, x_n)$ satisfying the following equation:

$$\begin{aligned} &\int_Q \mu(d\xi) \int_{R^n} \xi_n(dx_1 \cdots dx_n) u(\xi, x_1, \dots, x_n) \\ &\quad = \int_{R^n} \lambda_n(dx_1 \cdots dx_n) \int_Q \mu^{x_1, \dots, x_n}(d\xi) u(\xi + \delta_{x_1} + \cdots + \delta_{x_n}, x_1, \dots, x_n) \end{aligned} \tag{6.5}$$

for any measurable function $u(\xi, x_1, \dots, x_n)$ on $Q \times R^n$. These Palm measures satisfy the recursive relation

$$\mu^{x_1, x_2, \dots, x_n} = (\mu^{x_1, x_2, \dots, x_{n-1}})^{x_n} \quad \lambda_n\text{-a.e.}(x_1, \dots, x_n). \tag{6.6}$$

The following is a well-known fact which gives an intuitive picture to Palm measures.

Lemma 6.3. *Let λ_1 be a nonnegative nonatomic Radon measure and let μ be a probability measure on Q with intensity λ_1. Suppose that*

$$\int_{\xi(U) \geqslant 2} \xi(U) \mu(d\xi) = o(\lambda_1(U)) \tag{6.7}$$

for open sets as $U \to \{x\}$ for λ_1-a.e. x. Then the Palm measure μ^x is the limit of the conditional probability subject to the condition that there exists a particle in a neighborhood U of x as U shrinks to $\{x\}$. Precisely, for any bounded continuous function F,

$$\mu(F \mid \xi(U) > 0) \to \int_Q F(\xi + \delta_x) \mu^x(d\xi) \tag{6.8}$$

as $U \to \{x\}$ for λ_1-a.e. x. Moreover, if the nth correlation measures λ_n exists

$$\mu(F \mid \xi(U_i) > 0 \text{ for } 1 \leqslant i \leqslant n) \to \int_Q F(\xi + \delta_{x_1} + \cdots + \delta_{x_n}) \mu^{x_1, \dots, x_n}(d\xi) \tag{6.9}$$

as $U_i \to \{x_i\}$ $(1 \leqslant i \leqslant n)$ for λ_n-a.e. $(x_1, \dots, x_n)$.

Also it may be worthy to notice here that the spacing distribution is given by Palm measures when $R = \mathbb{R}^1$. Let $\theta(\xi)$ be the distance between the two particles in ξ that are the nearest and the second nearest to the origin 0 among those located on $[0, \infty)$.

Assume for simplicity the probability measure μ is translation invariant. Then there hold the equalities

$$\mu(\theta(\xi) > t) = \mu^0(\xi(0,t] = 0) = \frac{\partial}{\partial x}\mu(\xi(x,t] = 0)|_{x=0}. \tag{6.10}$$

Next lemma shows the relationship between correlation functions of μ and those of its Palm measures.

Lemma 6.4. *Let μ be a point process over R and fix a Radon measure λ on R. Assume that μ admits all the correlation functions $\{\rho_n\}_{n\geqslant 1}$ (with respect to λ). Then for $m \geqslant 1$ and for λ_m-a.e. $(x_1, \dots, x_m)$ the Palm measure $\mu^{x_1,\dots,x_m}$ admits all the correlation functions $\{\rho_n^{x_1,\dots,x_m}\}_{n\geqslant 1}$ (with respect to λ) and*

$$\rho_m(x_1,\dots,x_m)\cdot\rho_n^{x_1,\dots,x_m}(y_1,\dots,y_n) = \rho_{m+n}(x_1,\dots,x_m,y_1,\dots,y_n) \tag{6.11}$$

holds for $\lambda^{\otimes n}$-a.e. $(y_1,\dots,y_n)$.

Proof. We only give a formal proof. A rigorous proof can be easily done by induction on n and m keeping in mind the definition of ξ_n's. Let f, g be any bounded measurable nonnegative functions with compact support. Then,

$$\begin{aligned}
&\int_R \rho_1(x)\lambda(dx)g(x)e^{-f(x)}\int_Q \mu^x(d\xi)e^{-\langle\xi,f\rangle}\\
&\quad = \int_Q \mu(d\xi)\langle\xi,g\rangle e^{-\langle\xi,f\rangle}\\
&\quad = -\frac{d}{dt}\bigg|_{t=0}\int_Q \mu(d\xi)e^{-\langle\xi,f+tg\rangle}\\
&\quad = -\frac{d}{dt}\bigg|_{t=0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{R^n}\rho_n(x_1,\dots,x_n)\prod_{i=1}^{n}(1-e^{-f(x_i)-tg(x_i)})\lambda^{\otimes n}(dx_1\cdots dx_n)\\
&\quad = \int_R \lambda(dx_1)g(x_1)e^{-f(x_1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{R^n}\rho_{n+1}(x_1,x_2,\dots,x_{n+1})\\
&\qquad \times\prod_{i=2}^{n+1}(1-e^{-f(x_i)})\lambda^{\otimes n}(dx_2\cdots dx_{n+1}).
\end{aligned} \tag{6.12}$$

Hence, the correlation function $\rho_n^{x_1}$ of μ^{x_1} exists and is given by

$$\rho_n^{x_1}(y_1,\dots,y_n) = \frac{1}{\rho_1(x_1)}\rho_{n+1}(x_1,y_1,\dots,y_n). \tag{6.13}$$

Similarly, we can obtain (6.1) for $m \geqslant 2$. $\square$

6.2. Palm measures of fermion processes

As is mentioned in the introduction, the class of fermion processes is closed under the operation of taking Palm measures.

Theorem 6.5. *If $\mu_{-1,K}$ is the fermion process associated with operator K and if we denote its intensity by λ_1, then for λ_1-almost every x_0 the Palm measure $\mu^{x_0}_{-1,K}$ coincides with the fermion process associated with the operator K^{x_0} defined by*

$$K^{x_0}(x,y) = \frac{1}{K(x_0,x_0)} \det \begin{pmatrix} K(x,y) & K(x,x_0) \\ K(x_0,y) & K(x_0,x_0) \end{pmatrix} \tag{6.14}$$

whenever $K(x_0,x_0) > 0$.

Proof. Assume $K(x_0,x_0) > 0$ and show that K^{x_0} satisfies Condition A. In fact, $K^{x_0} \leqslant I$ because $K^{x_0} = K - K(x_0,\cdot) \otimes K(\cdot,x_0)/K(x_0,x_0) \leqslant K$. To see $K^{x_0} \geqslant 0$, one may apply to the eigenexpansion $K_\Lambda = \sum \kappa_n \varphi_n \otimes \varphi_n$ for any compact $\Lambda \subset R$. Then

$$\begin{aligned} \langle K^{x_0}_\Lambda f, f \rangle &= \sum_{n=1}^{\infty} \kappa_n \langle \varphi_n, f \rangle^2 - \frac{\left(\sum_{n=1}^{\infty} \kappa_n \varphi_n(x_0) \langle \varphi_n, f \rangle\right)^2}{\sum_{n=1}^{\infty} \kappa_n \varphi_n(x_0)^2} \\ &\geqslant 0. \end{aligned} \tag{6.15}$$

Hence $O \leqslant K^{x_0} \leqslant I$ and $K^{x_0}_\Lambda$ is of trace class for each compact $\Lambda \subset R$. Finally, it follows from Lemma 6.4 that

$$\rho^{x_0}_{n,-1,K}(x_1, \ldots, x_n) = \frac{1}{\rho_{1,-1,K}(x_0)} \rho_{n+1,-1,K}(x_0, x_1, \ldots, x_n). \tag{6.16}$$

On the other hand, it is immediate to see

$$\det(K^{x_0}(x_i,x_j))_{i,j=1}^n = \frac{1}{K(x_0,x_0)} \det(K(x_i,x_j))_{i,j=0}^n. \tag{6.17}$$

Hence,

$$\rho^{x_0}_{n,-1,K}(x_1, \ldots, x_n) = \det(K^{x_0}(x_i,x_j))_{i,j=1}^n. \tag{6.18}$$

Consequently, $\mu^{x_0}_{-1,K}$ is the fermion process associated with K^{x_0}. $\square$

By induction we have the following:

Corollary 6.6. *For each $n \geqslant 2$ the Palm measure $\mu_{-1,K}^{x_1,\dots,x_n}$ is associated with the integral kernel $K^{x_1,\dots,x_n}$ given by*

$$K^{x_1,\dots,x_n}(x,y) = (\det(K(x_i,x_j))_{i,j=1}^n)^{-1} \times \det \begin{pmatrix} K(x,y) & K(x,x_1) & \cdots & K(x,x_n) \\ K(x_1,y) & K(x_1,x_1) & \cdots & K(x_1,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,y) & K(x_n,x_1) & \cdots & K(x_n,x_n) \end{pmatrix} \tag{6.19}$$

for λ_n-a.e. $(x_1,\dots,x_n)$, where λ_n is the nth correlation measure of $\mu_{-1,K}$.

The following formula may be interesting in itself.

Example 6.7 (Kobayashi [16]). Let $R = \mathbb{R}^1$ and λ be the Lebesgue measure on it, and $K(x,y)$ be the resolvent kernel of a one-dimensional diffusion process or, the Green function for a Sturm–Liouville equation. Write

$$K(x,y) = \begin{cases} u(x)v(y) & \text{if } x \leqslant y, \\ u(y)v(x) & \text{if } x \geqslant y. \end{cases} \tag{6.20}$$

Then

$$\begin{aligned} &\det(K(x_i,x_j))_{i,j=1}^n \\ &\quad = K(x_1^*,x_1^*)K^{x_1^*}(x_2^*,x_2^*)K^{x_2^*}(x_3^*,x_3^*)\cdots K^{x_{n-1}^*}(x_n^*,x_n^*) \\ &\quad = K(x_n^*,x_n^*)K^{x_n^*}(x_{n-1}^*,x_{n-1}^*)K^{x_{n-1}^*}(x_{n-2}^*,x_{n-2}^*)\cdots K^{x_2^*}(x_1^*,x_1^*), \end{aligned} \tag{6.21}$$

where $x_1^* < \cdots < x_n^*$ is the rearrangement of $x_1,\dots,x_n$.

Proof. Let $a_i, b_i, i = 1,\dots,n$ be complex numbers. Then it is well known that

$$\det(a_{\min\{i,j\}}b_{\max\{i,j\}})_{i,j=1}^n = a_1 \cdot \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \cdot \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} \cdot \cdots \cdot \begin{vmatrix} b_{n-1} & b_n \\ a_{n-1} & a_n \end{vmatrix} \cdot b_n. \tag{6.22}$$

Hence (6.21) follows from the definition of $K^x(\cdot,\cdot)$. $\square$

Remark 6.8. Relation (6.21) shows that the spacings, i.e. the distances between nearest neighboring particles in ξ, are independent under $\mu_{-1,K}$ (cf. [9]). The converse is also true (cf. [37]).

6.3. Palm measures of boson processes: the case of nonnegative kernel

Recall that a point process μ is said to be infinitely divisible if for any $n \in \mathbb{N}$ there exists a point process ν_n so that μ is expressed by the n-fold convolution product of ν_n.

Theorem 6.9. *Assume Condition* B. *Then for any* $\alpha > 0$ *there exists a unique probability measure* $\mu_{\alpha,K}$ *such that*

$$\int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle) = \mathrm{Det}(I + \alpha K_\varphi)^{-1/\alpha}, \tag{6.23}$$

where $\varphi = 1 - e^{-f}$ *and* f *is a nonnegative measurable function with compact support. Moreover,* $\mu_{\alpha,K}$ *is always infinitely divisible.*

Proof. Assume Condition B. First we note that if the kernel function of the operator J_α is nonnegative then the operator $J_\alpha[\Lambda]$ also has nonnegative kernel. Indeed, as is easily seen by formula (3.36), we have

$$J_\alpha[\Lambda] = (J_\alpha)_\Lambda + \alpha (J_\alpha)_{\Lambda\Lambda^c}(I - \alpha (J_\alpha)_{\Lambda^c})^{-1}(J_\alpha)_{\Lambda^c\Lambda} \tag{6.24}$$

for any compact set $\Lambda \subset R$. Since $\det_\alpha(J_\alpha[\Lambda])$ is nonnegative for $\alpha > 0$, it follows from (2.16) and (3.2) that $\mathrm{Det}(I + \alpha K_\Lambda)^{-1/\alpha}$ is nonnegative and hence the density functions $\sigma_{\Lambda,\alpha,K}$ on Λ^n defined in (3.8) and (3.9) are nonnegative. Consequently, one can obtain the unique probability measure $\mu_{\alpha,K}$ satisfying (6.23).

Obviously, if α and K satisfy Condition B, so do $n\alpha$ and K/n for any $n \in \mathbb{N}$. Hence $\mu_{n\alpha,K/n}$ is also a probability measure and then the Laplace transform of $\mu_{\alpha,K}$ is equal to the nth power of the Laplace transform of $\mu_{n\alpha,K/n}$. Consequently, $\mu_{\alpha,K}$ is infinitely divisible. $\square$

Remark 6.10. If $\mu_{\alpha,K}$ is infinitely divisible, then the restriction $\mu_{\Lambda,\alpha,K}$ to the subspace $Q(\Lambda)$ is also infinitely divisible for each compact set Λ and we obtain the following representation by the Lévy measure (cf. [9,17]):

$$\begin{aligned}\int_Q &\mu_{\Lambda,\alpha,K}(d\xi) e^{-\langle \xi, f \rangle} \\ &= \exp\left(- \sum_{n=1}^\infty \frac{\alpha^{n-1}}{n} \int_{\Lambda^n} (1 - e^{-\sum_{i=1}^n f(x_i)}) \eta^\Lambda_{n,\alpha,K}(dx_1 \cdots dx_n) \right),\end{aligned} \tag{6.25}$$

where

$$\eta^\Lambda_{n,\alpha,K}(dx_1 \cdots dx_n) = \prod_{i=1}^n J_\alpha[\Lambda](x_i, x_{i+1}) \lambda^{\otimes n}(dx_1 \cdots dx_n) \tag{6.26}$$

with $x_{n+1} = x_1$. Indeed, if $\operatorname{supp} f \subset \Lambda$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\log \operatorname{Det}(I + \alpha(1 - e^{-f})K_\Lambda) \\
&\quad = -\frac{1}{\alpha}\log \operatorname{Det}(I - \alpha J_\alpha[\Lambda]) + \frac{1}{\alpha}\log \operatorname{Det}(I - \alpha e^{-f} J_\alpha[\Lambda]) \\
&\quad = \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n}[\operatorname{Tr}(J_\alpha[\Lambda])^n - \operatorname{Tr}(e^{-f} J_\alpha[\Lambda])^n] \\
&\quad = \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n}\int_{\Lambda^n}(1 - e^{-\sum_{i=1}^n f(x_i)})\eta^\Lambda_{n,\alpha,K}(dx_1 \cdots dx_n) \qquad (6.27)
\end{aligned}$$

by using (3.2) and the expansion formula (2.10) in Lemma 2.1.

Remark 6.11. When the underlying space R is a finite set, Griffiths and Milne [12] already discussed the necessary and sufficient condition on the matrix K for the infinite divisibility.

As was mentioned in Remark 6.2, the Palm measure of a Poisson random field Π is given by

$$\Pi^x = \Pi \text{ for } \lambda\text{-a.e. } x \qquad (6.28)$$

and that relation (6.28) gives a characterization of the Poisson random fields. This implies that the existence of a particle at x does not affect the location of other particles in the Poisson random fields while the next theorem indicates that the existence of a particle at x increases the number of particles in the boson random fields.

Theorem 6.12. *Let $\mu_{\alpha,K}$ be the point process given in Theorem 6.9 above. If we denote the intensity of $\mu_{\alpha,K}$ by λ_1, then for λ_1-a.e. x there exists a probability measure $\nu_{x,\alpha,K}$ on $\{\xi \in Q;\ \xi(R) < \infty\}$ such that the Palm measure $\mu^x_{\alpha,K}$ is given by the convolution*

$$\mu^x_{\alpha,K} = \mu_{\alpha,K} * \nu_{x,\alpha,K}. \qquad (6.29)$$

Proof. Let f, g be nonnegative measurable functions on R with compact support contained in a compact set Λ. Then

$$\begin{aligned}
&\int_R \lambda(dx)K(x,x)g(x)e^{-f(x)}\int_Q \mu^x_{\alpha,K}(d\xi)e^{-\langle \xi, f\rangle} \\
&\quad = \int_Q \mu_{\alpha,K}(d\xi)\int_R \xi(dx)g(x)e^{-\langle \xi, f\rangle}
\end{aligned}$$

$$= -\frac{d}{dt}\bigg|_{t=0} \int_Q \mu_{\alpha,K}(d\xi) e^{-\langle \xi, f+tg \rangle}$$

$$= -\frac{d}{dt}\bigg|_{t=0} \mathrm{Det}(I + \alpha(1 - e^{-f-tg})K_\Lambda)^{-1/\alpha}$$

$$= \mathrm{Det}(I + \alpha(1 - e^{-f})K_\Lambda)^{-1/\alpha} \,\mathrm{Tr}(ge^{-f}K_\Lambda(I + \alpha(1 - e^{-f})K_\Lambda)^{-1})$$

$$= \int_R \lambda(dx) K(x,x) g(x) e^{-f(x)} \{K_\Lambda(I + \alpha(1 - e^{-f})K_\Lambda)^{-1}\}(x,x)$$

$$\times \mathrm{Det}(I + \alpha(1 - e^{-f})K_\Lambda)^{-1/\alpha}. \tag{6.30}$$

Hence,

$$\int_Q \mu^x_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle)$$

$$= \{K_\Lambda(I + \alpha(1 - e^{-f})K_\Lambda)^{-1}\}(x,x) \cdot \int_Q \mu_{\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle). \tag{6.31}$$

Now we recall that the kernel function $J_\alpha[\Lambda]$ is nonnegative under Condition B as was seen in the proof of Theorem 6.9. Consequently, if $x \in \Lambda$ and $\mathrm{supp} f \subset \Lambda$, then

$$K_\Lambda(I + \alpha(1 - e^{-f})K_\Lambda)^{-1} = J_\alpha[\Lambda](I - \alpha e^{-f} J_\alpha[\Lambda])^{-1}$$

$$= \sum_{n=0}^{\infty} \alpha^n J_\alpha[\Lambda](e^{-f} J_\alpha[\Lambda])^n \tag{6.32}$$

and so one can define a probability measure by the formula

$$\int_Q \nu_{x,\alpha,K}(d\xi) \exp(-\langle \xi, f \rangle)$$

$$= \sum_{n=0}^{\infty} \alpha^n J_\alpha[\Lambda](e^{-f} J_\alpha[\Lambda])^n(x,x)$$

$$= \sum_{n=0}^{\infty} \alpha^n \int_{\Lambda^n} J_\alpha[\Lambda](x, x_1) \cdots J_\alpha[\Lambda](x_n, x) e^{-\sum_{i=1}^n f(x_i)} \lambda^{\otimes n}(dx_1 \cdots dx_n) \tag{6.33}$$

for any measurable function f of compact support. $\square$

6.4. Boson point processes and Gaussian random fields

The boson process and boson-like process ($\alpha = 1/m, m \in \mathbb{N}$) can be regarded as Cox processes [9]. It is well known that symmetric nonnegative definite Hilbert–Schmidt operators correspond to centered Gaussian random fields whose covariance are the given operators. In particular, under our Condition A there exists a Gaussian random field $X^\Lambda(x)$ on Λ for a symmetric integral operator K with continuous kernel and a compact subset Λ of R since K_Λ is then a Hilbert–Schmidt operator. It is not difficult to see that the family $\{X^\Lambda(x);\ x\in\Lambda\}$, Λ being a compact subset, satisfies the consistency condition and so there exists a Gaussian random field $X(x)$ on R with mean 0 such that $X(x)$ is locally square integrable with respect to λ and satisfies

$$E\left[\int_\Lambda X(x)^2\lambda(dx)\right] = \int_\Lambda K(x,x)\lambda(dx) < \infty \tag{6.34}$$

for each compact subset Λ of R and

$$E[X(x)X(y)] = K(x,y) \quad \text{for } \lambda\otimes\lambda\text{-a.e. } (x,y). \tag{6.35}$$

Thus we can consider the Poisson random field Π_{X^2} over R with random intensity $X(x)^2\lambda(dx)$. Then, it is immediate to see

$$\begin{aligned} E\left[\int_Q \Pi_{X^2}(d\xi)\exp(-\langle\xi,f\rangle)\right] &= E\left[\exp\left(-\int_R (1-e^{-f(x)})X(x)^2\lambda(dx)\right)\right] \\ &= \mathrm{Det}(I+2(1-e^{-f})K)^{-1/2}. \end{aligned} \tag{6.36}$$

Thus, the Poisson point process with random intensity $X^2\lambda$ gives us the probability measure $\mu_{2,K}$. The Boson point process associated with the integral operator K is given by the convolution of two independent copies $\mu_{2,K/2}$ or equivalently the Poisson point process with random intensity $(X^2+Y^2)\cdot\lambda$ where X and Y are independent copies of Gaussian random fields defined above from $K/2$. This construction brings us an extra bonus.

Theorem 6.13. *Assume Condition* A *as in the Theorem* 1.2. *Then for* $\alpha\in\{2/m;\ m\in\mathbb{N}\}$ *there exists a unique probability measure* $\mu_{\alpha,K}$ *such that* (1.7) *holds.*

Proof. We have already got a probability measure $\mu_{2,K}$ as above. The probability measure $\mu_{2/m,K}$ is nothing but the m-fold convolution of $\mu_{2,K/m}$. □

Remark 6.14. Dynkin gave an integration by parts formula for Gaussian random fields in [10]. The following special case is called Dynkin's isomorphism theorem in [1]: let $X = \{X(x)\}_{x\in R}$ be a Gaussian random field on R. Then there exists an independent random variable $L(x)$ such that

$$E\left[F(X^2)\frac{X(x)^2}{E[X(x)^2]}\right] = E[F(X^2+L(x))]. \tag{6.37}$$

The random variable $L(x)$ is known to be an occupation field of a certain Markov process on R starting at x and killed at x. In our context, this formula can be understood as a restatement of formula (6.29) in the case where $\alpha = 2$ and K is a Green operator. In particular,

$$\nu_{x,\alpha,K} = E[\Pi_{L(x)}]. \tag{6.38}$$

The fact that all the correlation functions of $\mu_{\alpha,K}$ for $\alpha \in \{2/m;\ m \in \mathbb{N}\}$ are nonnegative leads us to the following conclusion.

Corollary 6.15. *Let $\alpha \in \{2/m;\ m \in \mathbb{N}\}$. Then $\det_\alpha A$ is nonnegative whenever A is a nonnegative definite square matrix.*

Now expand the term $e^{-\langle \xi, f \rangle}$ in the left-hand side of (6.36) according to (4.2). Then one finds

$$E\left[\int_Q \Pi_{X^2}(d\xi) e^{-\langle \xi, f \rangle}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} E\left[\int_Q \Pi_{X^2}(d\xi) \langle \xi_n, \varphi_n \rangle\right], \tag{6.39}$$

where $\varphi_n(x_1, \ldots, x_n) = \prod_{i=1}^n (1 - e^{-f(x_i)})$ as in Section 4. Since Π_{X^2} is a Poisson point process with intensity $X(x)^2 \lambda(dx)$, we have

$$\int_Q \Pi_{X^2}(d\xi) \langle \xi_n, \varphi_n \rangle = \int_{R^n} X(x_1)^2 \cdots X(x_n)^2 \varphi_n(x_1, \ldots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n). \tag{6.40}$$

On the other hand, we know by Theorem 2.4 that if $||\varphi||_\infty ||2K|| < 1$

$$\begin{aligned} &\mathrm{Det}(I + 2(1 - e^{-f})K))^{-1/2} \\ &\quad = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{R^n} \det_2(K(x_i, x_j))_{i,j=1}^n \varphi_n(x_1, \ldots, x_n) \lambda^{\otimes}(dx_1 \cdots dx_n). \end{aligned} \tag{6.41}$$

Consequently,

$$\det{}_2(K(x_i, x_j))_{i,\ j=1}^n = E[X(x_1)^2 \cdots X(x_n)^2] \quad \lambda^{\otimes n}\text{-a.s.} \tag{6.42}$$

for each $n \geqslant 1$.

Similarly, we can obtain a representation of $\det_\alpha A$ for $\alpha = 2/m$, $m \in \mathbb{N}$, by Gaussian integrals. In particular, we have the following:

Corollary 6.16. *Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric nonnegative definite matrix and $Z = (Z_i)_{i=1,2,\ldots,n}$ be a Gaussian random variable with mean 0 and covariance A. Then the α-determinant for $\alpha = 2$ can be expressed as follows*:

$$\det_2 A = E[Z_1^2\ \cdots\ Z_n^2]. \tag{6.43}$$

Another, direct proof can be given by differentiating $\det(I + A)^{-1/2}$ repeatedly in a suitable manner [32].

6.5. A statistical–mechanical aspect

So far we constructed the random point field $\mu_{\alpha,K}$ starting from K and showed that the density $\sigma_{\Lambda,\alpha,K}$ is given in terms of the operator $J_\alpha[\Lambda] = K_\Lambda(I + \alpha K_\Lambda)^{-1}$. But, if one wants to interpret $\mu_{\alpha,K}$ as an object of statistical mechanics, it is natural to start from the operator $J_\alpha = K(I + \alpha K)^{-1}$. The operator J is the quasi-inverse of K in the sense that $(I - \alpha J_\alpha)(I + \alpha K) = I$ and its existence should be assumed if $\alpha < 0$.

Let H be a Hamiltonian operator and N be the number operator both realized on a L^2-space $L^2(R, \lambda)$. It may be quite natural to assume that the operator

$$J = e^{-\beta(H-\zeta N)} \quad (\beta > 0, \zeta \in \mathbb{R}) \tag{6.44}$$

is a symmetric operator and furthermore J may be assumed to be nonnegative definite. Assume, in addition, $\mathrm{Spec}(J) \subset [0, 1/|\alpha|)$ if $\alpha > 0$. Under so-called α-statistics the grand canonical partition function is given by an infinite product using eigenvalues. We may consider

$$Z_\alpha(\Lambda) = \mathrm{Det}(I - \alpha J_\Lambda)^{-1/\alpha} = \prod_{n=1}^{\infty} (1 - \alpha E_n(\Lambda))^{-1/\alpha}, \tag{6.45}$$

where J_Λ is the restriction of J to $L^2(\Lambda, d\lambda)$ and $E_n(\Lambda)$, $n \geqslant 1$ are the eigenvalues of J_Λ. The α-statistics is the fermion and the boson statistics if $\alpha = -1$ and 1, respectively. If we want to realize the grand canonical ensemble $\mu_\alpha^{(\Lambda)}$ as a random point field, its Laplace transform will be

$$\int_{Q(\Lambda)} \mu_\alpha^{(\Lambda)}(d\xi) \exp(-\langle \xi, f \rangle) = \frac{1}{Z_\alpha(\Lambda)} \mathrm{Det}(I - \alpha (J_\Lambda)_{e^{-f}})^{-1/\alpha}. \tag{6.46}$$

Applying Proposition 3.11 to (6.46), we obtain the following second construction of $\mu_{\alpha,K}$ starting from J.

Theorem 6.17. *Let $\alpha \in \{-1/m;\ m \in \mathbb{N}\} \cup \{2/m;\ m \in \mathbb{N}\}$ and J be a bounded symmetric integral operator with continuous kernel $J(x, y)$. Assume J is nonnegative definite and, in addition, $||\alpha J|| < 1$ if $\alpha > 0$. For a compact subset Λ of R define a probability measure $\mu_\alpha^{(\Lambda)}$ by*

$$\mu_\alpha^{(\Lambda)}(dx_1 \cdots dx_n) = \frac{1}{Z_\alpha(\Lambda)} \det_\alpha (J(x_i, x_j))_{i,j=1}^n \lambda^{\otimes n}(dx_1 \cdots dx_n) \tag{6.47}$$

on each Λ^n. *Then* $\mu_\alpha^{(\Lambda)}$ *satisfies* (6.46) *and converges as* Λ *tends to* R *to the probability measure* $\mu_{\alpha,K}$ *constructed in Theorem* 1.2 *with* $K = J(I-\alpha J)^{-1}$. *In other words*,

$$\begin{aligned}\int_{Q(\Lambda)} \mu_\alpha^{(\Lambda)}(d\xi)\exp(-\langle \xi, f\rangle) &= \frac{1}{Z_\alpha(\Lambda)} \mathrm{Det}(I-\alpha e^{-f}J_\Lambda)^{-1/\alpha} \\ &\to \mathrm{Det}(I+\alpha K_\varphi)^{-1/\alpha} \\ &= \int_Q \mu_{\alpha,K}(d\xi)\exp(-\langle \xi, f\rangle) \end{aligned} \tag{6.48}$$

as Λ *tends to* R *for each nonnegative measurable function* f *with compact support*.

Proof. It is obvious that $\mu_\alpha^{(\Lambda)}$ satisfies (6.46). Set

$$K_\alpha[\Lambda] = J_\Lambda(I-\alpha J_\Lambda)^{-1} = ((I-\alpha J_\Lambda)^{-1} - I)/\alpha. \tag{6.49}$$

Then we have

$$\begin{aligned}\frac{\mathrm{Det}(I-\alpha e^{-f}J_\Lambda)}{\mathrm{Det}(I-\alpha J_\Lambda)} &= \mathrm{Det}((I+\alpha K_\alpha[\Lambda]) - \alpha e^{-f}K_\alpha[\Lambda]) \\ &= \mathrm{Det}(I+\alpha\varphi K_\alpha[\Lambda]). \end{aligned} \tag{6.50}$$

By using (3.38) in Remark 3.8 we obtain

$$I_\Lambda + \alpha(K_\alpha[\Lambda'])_\Lambda = \{(I-\alpha J_{\Lambda'})^{-1}\}_\Lambda \geqslant (I_\Lambda - \alpha J_\Lambda)^{-1} = I_\Lambda + \alpha(K_\alpha[\Lambda])_\Lambda. \tag{6.51}$$

Thus, for any $f \in L^2(R,\lambda)$,

$$\begin{aligned}\langle (\alpha K_\alpha[\Lambda''])_\Lambda f, f\rangle &= \langle (\alpha K_\alpha[\Lambda''])_{\Lambda'} f_\Lambda, f_\Lambda\rangle \\ &\geqslant \langle (\alpha K_\alpha[\Lambda'])_{\Lambda'} f_\Lambda, f_\Lambda\rangle \\ &= \langle (\alpha K_\alpha[\Lambda'])_\Lambda f, f\rangle \end{aligned} \tag{6.52}$$

whenever $\Lambda \subset \Lambda' \subset \Lambda''$. Hence, $(\alpha K_\alpha[\Lambda'])_\Lambda$ is nondecreasing in Λ' in the sense of quadratic forms and converges strongly to

$$\{I_\Lambda - \alpha J_\Lambda - \alpha^2 J_{\Lambda,\Lambda^c}(I_{\Lambda^c} - \alpha J_{\Lambda^c})^{-1} J_{\Lambda^c,\Lambda}\}^{-1} - I_\Lambda = \alpha K_\Lambda. \tag{6.53}$$

Consequently, we can apply Proposition 3.11 and obtain

$$\mathrm{Det}(I+\alpha\sqrt{\varphi}(K_\alpha[\Lambda'])_\Lambda\sqrt{\varphi}) \to \mathrm{Det}(I+\alpha\sqrt{\varphi}K_\Lambda\sqrt{\varphi}) \tag{6.54}$$

as $\Lambda' \to R$ when $\mathrm{supp} f \subset \Lambda$. $\square$

7. On α-determinant

7.1. Conjecture and partial results

In Section 2 we encountered the function $\det_\alpha A$ of a matrix A defined by

$$\det_\alpha A = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{7.1}$$

for an n by n matrix $A = (a_{ij})_{i,j=1}^n$ where $\mathfrak{S}_n$ is the symmetric group of order n and $\nu(\sigma)$ is the number of the cycles which consist of σ. The existence problem of random point fields associated with $\mathrm{Det}(I + K_\varphi)^{-1/\alpha}$ was equivalent to the nonnegativity problem of $\det_\alpha A$ for all nonnegative definite matrices. The nonnegativity is trivial if the entries of A is nonnegative and $\alpha > 0$ even if A is not symmetric matrix. For a nonnegative definite matrix we have proved the nonnegativity for $\alpha \in \{2/m;\ m \in \mathbb{N}\} \cup \{-1/m;\ m \in \mathbb{N}\}$ by the probabilistic construction given in Section 3 and in Section 6, respectively. Besides, one can easily see that $\det_\alpha A \geqslant 0$ for small α's for each fixed matrix size. We strongly feel that the following is true.

Conjecture 7.1. *Let* $0 \leqslant \alpha \leqslant 2$. *Then* $\det_\alpha A$ *is nonnegative whenever* A *is a nonnegative definite square matrix.*

It is easy to see that Conjecture 7.1 for $\alpha < 0$ fails unless $\alpha \in \{-1/m;\ m \in \mathbb{N}\}$.

Conjecture 7.2. *Let* $\alpha > 2$. *Then there exists a matrix size* $n(\alpha)$ *such that the nonnegativity of* $\det_\alpha A$ *fails for some nonnegative definite square matrix* A *of size* n *if and only if* $n \geqslant n(\alpha)$.

Remark 7.3. (i) The usual q-analogue of determinants is defined by using the inversion number $\iota(\sigma)$ in place of $n - \nu(\sigma)$ where $0 \leqslant q \leqslant 1$ and $\iota(\sigma) = \#\{(i,j);\ 1 \leqslant i < j \leqslant n \text{ and } \sigma(i) > \sigma(j)\}$. The function $d(\sigma, \tau) = \iota(\sigma^{-1}\tau)$ is a distance in $\mathfrak{S}_n$ and the matrix $(q^{\iota(\sigma^{-1}\tau)})_{\sigma,\tau \in \mathfrak{S}_n}$ is nonnegative definite. Hence, this q-analogue is nonnegative if A is nonnegative definite [6]. But the matrix $(\alpha^{n-\nu(\sigma^{-1}\tau)})_{\sigma,\tau \in \mathfrak{S}_n}$ is not nonnegative definite in general for $0 < \alpha < 1$. Indeed, $c^{(\lambda)}(\alpha)$'s defined below in (7.4) are the eigenvalues of this matrix.

(ii) It is well known that there is one to one correspondence between the equivalence class of irreducible characters of $\mathfrak{S}_n$ and the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n. The following quantity is called immanant of A (cf. [13,20]):

$$\det_{\chi^{(\lambda)}} A = \sum_{\sigma \in \mathfrak{S}_n} \chi^{(\lambda)}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{7.2}$$

where $\chi^{(\lambda)}$ is the irreducible character which is associated with a partition λ. It is also known that $\det_{\chi^{(\lambda)}} A$ is nonnegative whenever A is nonnegative definite. Thus one can discuss point processes associated with immanants (cf. [8]).

(iii) It is not so difficult to see the following formula which seems to be well known among specialists [26]:

Let A be an n by n matrix. For any $\alpha \in \mathbb{R}$, we obtain

$$\det_\alpha A = \sum_\lambda \frac{1}{n!} (\dim \lambda) c^{(\lambda)}(\alpha) \det_{\chi^{(\lambda)}} A, \tag{7.3}$$

where $\dim \lambda$ is the dimension of an irreducible representation associated with a partition λ and we set

$$c^{(\lambda)}(\alpha) = \prod_{i=1}^{k} \prod_{j=1}^{\lambda_i} (1 + (j - i)\alpha). \tag{7.4}$$

The dimension $\dim \lambda$ is given by the formula

$$\dim \lambda = \frac{n!}{\ell_1! \cdots \ell_k!} \Delta(\ell_1, \ldots, \ell_k), \tag{7.5}$$

where $\ell_i = \lambda_i + k - i$ $(1 \leqslant i \leqslant k)$ and $\Delta(\ell_1, \ldots, \ell_k) = \prod_{i<j} (\ell_j - \ell_i)$ is the Vandermonde determinant.

Formula (7.3) gives a quick proof to the fact that $\det_\alpha$ is nonnegative for $\alpha \in \{\pm 1/m;\ m \in \mathbb{N}\} \cup [0, 1/n]$ if A is an n by n nonnegative definite matrix. But the fact that $\det_\alpha$ is nonnegative also for $\alpha \in \{2/m;\ m \in \mathbb{N}\}$ is rather mysterious and is difficult, at least to the authors, to be deduced from formula (7.3) with (7.4) and (7.5).

Acknowledgments

We express sincere thanks to the referee and Professor R. Lyons for their valuable comments on the manuscripts.

References

[1] R.J. Adler, An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes, Lecture Notes-Monograph Series, Vol. 12, Institute of Mathematical Statistics, Hayward, California, 1990.
[2] R. Bhatia, Matrix analysis, in: Graduate Texts in Mathematics, Vol. 169, Springer, Berlin, 1997.
[3] A. Borodin, A. Okounkov, G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. 13 (2000) 481–515.

[4] A. Borodin, G. Olshanski, Point processes and the infinite symmetric group part III: fermion point processes, available via http://xxx.lanl.gov/abs/math.RT/9804088.
[5] A. Borodin, G. Olshanski, Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, available via http://xxx.lanl.gov/abs/math.RT/0109194.
[6] M. Bożejko, R. Speicher, An example of a generalized Brownian motion, Comm. Math. Phys. 137 (1991) 519–531.
[7] O. Costin, J. Lebowitz, Gaussian fluctuation in random matrices, Phys. Rev. Lett. 75 (1995) 69–72.
[8] P. Diaconis, S.N. Evans, Immanants and finite point processes, In memory of Gian-Carlo Rota, J. Combin. Theory Ser. A 91 (2000) 305–321.
[9] D.J. Daley, D. Veres-Jones, An Introduction to the Theory of Point Processes, Springer, Berlin, 1988.
[10] E.B. Dynkin, Gaussian and non-Gaussian random fields associated with Markov processes, J. Funct. Anal. 55 (1984) 344–376.
[11] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Translations of Mathematical Monographs, Vol. 18, AMS, Providence, RI, 1969.
[12] R.C. Griffiths, R.K. Milne, A class of infinitely divisible multivariate negative binomial distributions, J. Multivariate Anal. 22 (1987) 13–23.
[13] G. James, Permanents, immanants, and determinants, Proc. Symp. Pure Math. 47 (1987) 431–436.
[14] K. Johansson, Discrete polynuclear growth and determinantal processes, available via http://xxx.lanl.gov/abs/math.PR/0206208.
[15] M. Kac, Toeplitz matrices, translation kernel and a related problem in probability theory, Duke Math. J. 21 (1954) 501–509.
[16] Y. Kobayashi, private communication.
[17] J. Kerstan, K. Matthes, J. Mecke, Infinite Divisible Point Processes, Nauka, Moscow, 1982.
[18] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I, Arch. Rationa. Mech. Anal. 59 (1975) 219–239.
[19] E.H. Lieb, Proofs of some conjectures on permanents, J. Math. Mech. 16 (1966) 127–134.
[20] D.E. Littlewood, The Theory of Group Characters and Matrix Representations of Groups, 2nd Edition, Oxford University Press, London, 1958.
[21] R. Lyons, Determinantal probability measures, preprint.
[22] R. Lyons, J.E. Steif, Stationary Determinantal processes: phase multiplicity, Bernoullicity, entropy, and domination, preprint.
[23] E. Lytvynov, Fermion and boson random point processes as particle distributions of infinite free Fermi and Bose gases of finite density, available via http://arXiv.org/abs/math-ph/0112006.
[24] O. Macchi, The coincidence approach to stochastic point processes, Adv. Appl. Probab. 7 (1975) 83–122.
[25] O. Macchi, The fermion process—a model of stochastic point process with repulsive points, Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes and of the Eighth European Meeting of Statisticians (Tech. Univ. Prague, Prague, 1974), Vol. A, pp. 391–398.
[26] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd Edition, Clarendon Press, Oxford, 1995.
[27] M.L. Mehta, Random Matrices, 2nd Edition, Academic Press, New York, 1991.
[28] H. Osada, Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions, Comm. Math. Phys. 176 (1996) 117–131.
[29] M. Prähofer, H. Spohn, Scale invariance of the PNG droplet and the Airy process, available via http://xxx.lanl.gov/abs/math.PR/0105240.
[30] T. Shirai, Y. Takahashi, Fermion process and Fredholm determinant, in: H.G.W. Begehr, R.P. Gilbert, J. Kajiwara (Eds.), Proceedings of the Second ISAAC Congress, Vol. 1, 1999, Kluwer Academic Publishers, Dordrecht, 2000, pp. 15–23.
[31] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants II: fermion shifts and their ergodic and Gibbs properties, Ann. Probab., to appear.
[32] T. Shirai, Y. Takahashi, Random point fields associated with Fermion, Boson and other statistics, Adv. Stud. Pure Math., to appear.

[33] T. Shirai, H.J. Yoo, Glauber dynamics for fermion point processes, Nagoya Math. J. 168 (2002) 139–166.
[34] B. Simon, Trace Ideals and their Applications, in: London Mathematical Society Lecture Note Series, Vol. 35, Cambridge University Press, Cambridge, 1977.
[35] A. Soshnikov, Gaussian fluctuation for the number of particles in Airy, Bessel, Sine and other determinantal random point fields, J. Statist. Phys. 100 (2000) 491–522.
[36] A. Soshnikov, The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities, Ann. Probab. 28 (2000) 1353–1370.
[37] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000) 923–975.
[38] A. Soshnikov, Gaussian limit for determinantal random point fields, Ann. Probab. 30 (2002) 171–187.
[39] H. Spohn, Interacting Brownian particles: a study of Dyson's model, in: G. Papanicalaou (Ed.), Hydrodynamic Behavior and Interacting Particle Systems, IMA Volumes in Mathematics and Its Applications, Vol. 9, Springer, New York, 1987, pp. 151–179.
[40] C.A. Tracy, H. Widom, Correlation functions, cluster functions and spacing distribution for random matrices, J. Stat. Phys. 92 (5/6) (1998) 809–835.
[41] D. Vere-Jones, A generalization of permanents and determinants, Linear Algebra Appl. 111 (1988) 119–124.
[42] A. Zygmund, Trigonometrical Series, Third ed., Vols. I and II combined, Cambridge Univ. Press, Cambridge, 2002.